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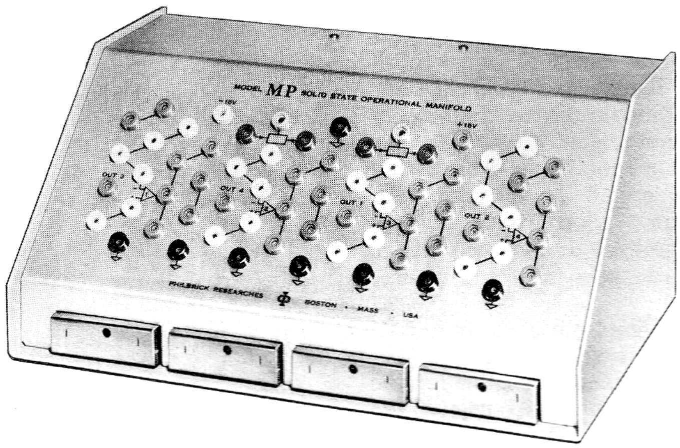
A MANIFOLD FOR MANIFOLD USES
(including those of indoctrination into operational techniques)



Ages ago, technologically speaking, Philbrick put out a small group of Operational Manifolds, having vacuum tube amplifiers and largely educational applications. Many engineers and experimenters who have since graduated to solid-state apparatus first learned the feedback art on these earlier machines. Certain of these people are now captains of industry, or otherwise important.

Now of course there are newer technologists who look on vacuum tubes as archaic, but who still could derive much comfort and convenience, in accurate instrumentation, from indoctrination in operational amplifiers. Hence the modern, transistorized version of our earlier manifolds, useful, as before, for practical as well as pedagogical Modelling, Measuring, Manipulating and much else.

The Model MP Operational Manifold, shown here, is elsewhere described in greater detail. It is self-powered at ± 15 vdc, and is intended for use in the ± 10 vdc range. Four standard Philbrick operational amplifiers plug in from the front and are adjustable in place, in the unlikely event this is needed. The sloping front panel, with its suggestive symbology, accommodates interconnections and circuit components which complete the feedback and feed-forward path of the desired computational structures.

The time scale over which this nest of amplifiers may be made to operate is from fractions of milliseconds to many seconds. Some might prefer to talk of frequencies from millicycles to many kilocycles per second. We deal amicably with both sorts of people, though we secretly enthuse over time-domain concepts and considerations. The available resolution in the voltage direction is just as impressive, since randomness in voltage and current may be kept to a few parts per million. This space-time smoothness and flexibility enables the putting of this tool to work in performing continuous algebra, the infinitesimal calculus (operational or conventional), and an endless variety of instrumental mathematics which stem therefrom. Nor must we forget the group of discrete and/or logical operations ordinarily thought of as digital. Lightning empiricism embraces all of this.

Furthermore the operations performed may be repetitive or "single-shot". The differential equations solved may be linear or nonlinear. The installation may be permanent, *pro tem*, or peripatetic. Indeed, the purpose may be indoctrination or instrumentation. Whatever your analog opportunity or task, Philbrick will enjoy showing how this new manifold may be adapted to it. If it is inappropriate, we shall be the first to point this out. If, on the other hand some other Philbrick product is indicated, including our very broad line of operational amplifiers and accessories, it will be our pleasure and privilege to indoctrinate you into its benefits and the techniques for its application.

NEW APPROACHES TO THE DESIGN OF ACTIVE LOW PASS FILTERS

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(Part II)

Operational amplifiers enable the designer to achieve extremely high "Q" (low damping ratio) filter characteristics, since inductors can be avoided. Such circuits can be used over a wide range of frequencies, approximately .01 cps to 100 kc. Furthermore, there are often cost and size advantages as well.

In the preceding issue (L.E. Vol. 13, No. 1 & 2) we discussed the design of low pass filter circuits. In this issue we discuss the remaining classical linear filter types: high pass, band pass, band reject, and all pass. In most cases the design methods indicated here build upon those presented in the first article.

Once the desired transfer function is selected, it can be programmed directly on an analog computor. The general purpose computor approach which was presented in the first article enables the user rapidly to modify the filter characteristic and thus empirically to optimize a design.

Special-purpose circuitry, employing less than half the number of amplifiers required when using a general purpose analog computor, can result in considerable economy. However, the ability to modify easily the transfer function is sacrificed. For special-purpose modelling the transfer function may be realized as a cascade of first and second order transfer functions. In this case, it is necessary to determine the roots of both the numerator (zeros) and the denominator (poles) polynomials. In this article we will show how conformal transformation of a factored low pass filter yields the high pass, band pass, or band reject filter characteristic in factored form. Then, using Figures 1-5, an amplifier circuit having the appropriate form may be selected for each factor. Each of these must be driven from a low impedance (voltage) source and provides a low impedance output.

Transformations

By applying a conformal transformation to a low pass filter characteristic it is possible to convert that characteristic to a high pass, band pass, or band reject characteristic having exactly the same gain and phase as the low pass at corresponding frequencies. Table 1 lists these transformations.

Figure 6 illustrates the gain and phase characteristics of all of the transformed filter types indicated in Table 1. Selected corresponding points are on each of the transformed gain and phase characteristics. Note that there is a discontinuity of $n\pi$ radians in some of the phase characteristics as the gain passes through zero.

For example, to convert a low pass filter characteristic having a nominal cut-off frequency, ω_C; and Heaviside operator, $p_l = \dfrac{d}{dt}$:

$$y = A\left\{\frac{p_l}{\omega_c}\right\} \cdot x$$

into a band pass filter characteristic having a lower cut-off frequency at ω_{CL} and upper cut-off frequency ω_{CU}, one employs the mapping relation:

$$\frac{p_l}{\omega_c} = \left(\frac{p}{\omega_o} + \frac{\omega_o}{p}\right)\frac{1}{2B}$$

where: $\omega_o = \sqrt{\omega_{CU} \cdot \omega_{CL}}$ "center frequency"

$B = \dfrac{\omega_{CU} - \omega_{CL}}{2\omega_o}$ "bandwidth to center frequency ratio"

p is the band pass Heaviside operator.

Thus achieving:

$$y = A\left\{\left(\frac{p}{\omega_o} + \frac{\omega_o}{p}\right)\frac{1}{2B}\right\} \cdot x$$

It is important to note that the low pass cut-off frequency ω_c can be arbitrarily defined, i.e. it is not necessarily the 3 db down or 45° phase lag frequency. The transformed band pass filter then has the same gain and phase at ω_{CU} and ω_{CL} as does the low pass filter at ω_c.

When $A\left\{\dfrac{p_l}{\omega_c}\right\}$ is known in factored form, each factor can be transformed separately. A first-order low pass factor transforms into a second-order band pass factor and a second order low pass factor transforms into two second order factors. These two second-order factors are of the band pass type if B < 1 (narrow pass band). When B ≥ 1 the factor having the lower natural frequency, $\dfrac{\omega_o}{\alpha}$, should be a high pass factor and the factor having the larger natural frequency should be a low pass factor. (α is defined subsequently)

The gain or phase at any frequency, ω, can be determined by first calculating the corresponding low pass frequency, ω_l, which for our band pass example is (see Table 1):

$$\frac{\omega_l}{\omega_c} = \left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\frac{1}{2B}$$

Then the gain or phase is determined by referring to plots or equations of the low pass characteristics at ω_l. If the low pass filter had an nth order Butterworth characteristic the gain could be determined from:

$$\left|A\left\{\frac{\omega_l}{\omega_c}\right\}\right|^2 = \frac{1}{1 + \left(\dfrac{\omega_l}{\omega_c}\right)^{2n}}$$

Table 1 Transformations of Low Pass Filters

	Low Pass Model	High Pass	Band Pass	Band Reject
Mapping Relation	$\frac{p_l}{\omega_c} =$	$\frac{\omega_o}{p}$	$\left(\frac{p}{\omega_o} + \frac{\omega_o}{p}\right)\frac{1}{2B}$	$\frac{2B}{\left(\frac{p}{\omega_o} + \frac{\omega_o}{p}\right)}$
Corresponding Frequencies	$\frac{\omega_l}{\omega_c} =$	$-\frac{\omega_o}{\omega}$	$\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\frac{1}{2B}$	$\frac{-2B}{\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)}$
Transfer Function First Order Model	$\frac{1}{1 + \frac{p_l}{\omega_L}} \rightarrow$	$\frac{\frac{p}{\omega_o'}}{1 + \frac{p}{\omega_o'}}$	$\frac{2B_1 \frac{p}{\omega_o}}{1 + 2B_1 \frac{p}{\omega_o} + \left(\frac{p}{\omega_o}\right)^2}$	$\frac{1 + \left(\frac{p}{\omega_o}\right)^2}{1 + 2B_1 \frac{p}{\omega_o} + \left(\frac{p}{\omega_o}\right)^2}$
Second Order Model	$\frac{1}{1 + 2\xi_L \frac{p_l}{\omega_L} + \left(\frac{p_l}{\omega_L}\right)^2} \rightarrow$	$\frac{\left(\frac{p}{\omega_o'}\right)^2}{1 + 2\xi_L \frac{p}{\omega_o'} + \left(\frac{p}{\omega_o'}\right)^2}$	$\left[\frac{2B_1\left(\frac{p}{\alpha\omega_o}\right)}{1 + 2\xi\left(\frac{p}{\alpha\omega_o}\right) + \left(\frac{p}{\alpha\omega_o}\right)^2}\right] \cdot \left[\frac{2B_1\left(\frac{\alpha p}{\omega_o}\right)}{1 + 2\xi\left(\frac{\alpha p}{\omega_o}\right) + \left(\frac{\alpha p}{\omega_o}\right)^2}\right]$	$\left[\frac{1 + \left(\frac{p}{\omega_o}\right)^2}{1 + 2\xi\left(\frac{p}{\alpha\omega_o}\right) + \left(\frac{p}{\alpha\omega_o}\right)^2}\right] \cdot \left[\frac{1 + \left(\frac{p}{\omega_o}\right)^2}{1 + 2\xi\left(\frac{\alpha p}{\omega_o}\right) + \left(\frac{\alpha p}{\omega_o}\right)^2}\right]$
Definitions	ω_c cut off frequency ω_L natural frequency of 2nd order factor ξ_L damping ratio p_l low pass Heaviside operator ω_l low pass radian frequency	$\omega_o' = \frac{\omega_c}{\omega_L}\omega_o$	$B_1 = B\frac{\omega_L}{\omega}$ $\xi = \frac{2\xi_L B_1}{\alpha + \frac{1}{\alpha}}$ (Band Pass and Band Reject) $\alpha = \sqrt{\frac{B_1^2 + 1 + \sqrt{B_1^4 + 2(1 - 2\xi_L^2)B_1^2 + 1}}{2}} + \sqrt{\frac{B_1^2 - 1 + \sqrt{B_1^4 + 2(1 - 2\xi_L^2)B_1^2 + 1}}{2}}$ (Band Pass and Band Reject) $B = \frac{\omega_{CU} - \omega_{CL}}{2\omega_o}$ (Band Pass and Band Reject) $\omega_o = \sqrt{\omega_{CU}\omega_{CL}}$ (Band Pass and Band Reject)	$B_1 = B\frac{\omega_c}{\omega_L}$

Table 2 Low Pass Transfer Functions

Order	Butterworth	Paynter
1.	$\dfrac{1}{\left(1+\dfrac{p}{\omega_c}\right)}$	
2.	$\dfrac{1}{\left(1+\sqrt{2}\dfrac{p}{\omega_c}+\left(\dfrac{p}{\omega_c}\right)^2\right)}$	$\dfrac{1}{\left(1+3\dfrac{p}{\omega_c}+4\left(\dfrac{p}{\omega_c}\right)^2\right)}$
3.	$\dfrac{1}{\left(1+\dfrac{p}{\omega_c}\right)\left(1+\dfrac{p}{\omega_c}+\left(\dfrac{p}{\omega_c}\right)^2\right)}$	$\dfrac{1}{\left(1+2\dfrac{p}{\omega_c}\right)\left(1+1.2\left(\dfrac{p}{\omega_c}\right)+1.6\left(\dfrac{p}{\omega_c}\right)^2\right)}$
	$\omega_c = 3$ db frequency $\lvert A\rvert = .7$ when $\omega = \omega_c$ $\lvert A\rvert^2 = \dfrac{1}{1+\left(\dfrac{\omega}{\omega_c}\right)^{2n}}$	$\phi_{lag} \approx \pi \dfrac{\omega}{\omega_c}$

Low Pass Filters

Of the various types of low pass filters discussed in the first article, we believe the most useful to be the Butterworth and Paynter. The Butterworth filter is designed to be optimally flat in the pass band and to have a very sharp cut-off. However, its transient step response overshoots significantly.

The low pass Paynter filter has nearly linear phase shift with frequency in the pass band; hence its transient response characteristics are good. However, its amplitude vs. frequency characteristic is neither as flat as that of the Butterworth in the pass band nor has it as sharp a cut-off.

The transformations preserve the nature of the amplitude vs. frequency characteristic both in the pass band and in the stop band. Hence a transformed Butterworth filter has desirable gain vs. frequency characteristics. However, the transformed high pass, band pass, and band reject transient responses bear no obvious relationship to that of the corresponding low pass filter. Thus even though a low pass Paynter filter has a desirable transient response characteristic, the transformed Paynter filters do not necessarily have any useful transient response properties.

Table two lists some low order transfer functions for the low pass Butterworth and Paynter filters.

High Pass Filters

The high pass characteristic achieved by the mapping relation can be very effective at removing all frequency components below the cut-off frequency, ω_o, while passing all frequencies above the cut-off frequency. However, the resulting transient response does not resemble the original waveform if the original waveform has significant frequency content in the neighborhood of the cut-off frequency. Transient distortion results because these components in the pass band are little attenuated but have significant phase shift which is not even approximately proportional to frequency.

Low transient distortion in a high pass filter can be achieved only if the non-attenuated components have little phase shift. The filter type producing the least phase shift at the cut-off frequency is the first order lead:

$$\frac{\dfrac{p}{\omega_o}}{1+\dfrac{p}{\omega_o}}$$

This can be realized very effectively using the circuit of Figure 4a. Extremely low cut-off frequencies can be achieved (e.g. $R = 100$ Meg, $C = 10\ \mu F$
$\omega_o = .001$ rad/sec.)
with an amplifier having very high common mode input impedance and very low input current ($<10^{-9}$ amp.) such as the SP2A, P25A, or Q25AH.

Other high pass circuits are shown in Figure 4*b* and *c*. These can be used for the quadratic factors of high order filters. It must be remembered that the amplifier gain-bandwidth limitation ultimately limits the high-frequency pass band. Depending upon the amplifier used a small feedback capacitor may be necessary to achieve stability. These comments also apply to the band reject filter.

Wide Band Pass Filters

A bandpass filter has two "cut-off" frequencies. Below the lower, ω_{CL}, and above the upper, ω_{CU}, signal components are highly attenuated. Between the two cut-off frequencies the signal components are passed with nearly unity gain, and with phase shift varying significantly with frequency. To simplify the algebra it is convenient to define two new terms:

$$\omega_o = \sqrt{\omega_{CU}\cdot\omega_{CL}} \qquad \text{“center frequency”}$$

$$B = \frac{\omega_{CU}-\omega_{CL}}{2\omega_o} \qquad \text{“normalized bandwidth”}$$

We classify those band-pass filters having a

$\mathrm{B} \geqslant 1, \left(\dfrac{\omega_{CU}}{\omega_{CL}} > 6\right)$ as wide band and those with

$\mathrm{B} < 1, \left(\dfrac{\omega_{CU}}{\omega_{CL}} < 6\right)$ as narrow band.

The wide band pass filter is achieved by cascading a high pass filter with cut-off frequency at approximately ω_{CL} with a low pass filter with cut-off at approximately ω_{CU}. The transformation method of Table 1 yields a band pass characteristic which is symmetric about ω_o when plotted with logarithmic coordinates. A transformed first order low pass factor becomes:

$$\frac{1}{1+\frac{p_l}{\omega_c}} \rightarrow \frac{2B}{\alpha}\left[\frac{\alpha\frac{p}{\omega_o}}{1+\alpha\frac{p}{\omega_o}}\right]\left[\frac{1}{1+\frac{p}{\alpha\omega_o}}\right]$$

$$\alpha = B + \sqrt{B^2 - 1}$$

A transformed second order low pass factor becomes:

$$\frac{1}{1+2\xi_L\frac{p_l}{\omega_c}+\left(\frac{p_l}{\omega_c}\right)^2} \rightarrow \frac{2B}{\alpha}\left[\frac{\alpha\left(\frac{p}{\omega_o}\right)^2}{1+2\xi\left(\alpha\frac{p}{\omega_o}\right)+\left(\alpha\frac{p}{\omega_o}\right)^2}\right]\left[\frac{1}{1+2\xi\left(\frac{p}{\alpha\omega_o}\right)+\left(\frac{p}{\alpha\omega_o}\right)^2}\right]$$

where for $B > 2$, $\left(\frac{\omega_{CU}}{\omega_{CL}} > 20\right)$

$$\xi \approx \xi_L\left[\frac{1}{1+\frac{1}{2B^2}(1-\xi_L^2)}\right]$$

$$\frac{\alpha}{2B} \approx 1 + \frac{1}{4B^2}(1 - 2\xi_L^2)$$

When it is necessary to minimize transient distortion it is generally preferable to design separately the high pass and low pass sections of the wide band pass filter instead of using the low pass to band pass transformation. Thus one may choose to have a simple first order lead with break frequency at ω_1 cascaded with a third order Paynter low pass filter with characteristic frequency, ω_2 $\left(\phi_{Lag} = \pi\frac{\omega}{\omega_2}\right)$, provided $\omega_2 > 100\omega_1$.

$$A(p) = \left[\frac{\frac{p}{\omega_1}}{1+\frac{p}{\omega_1}}\right]\left[\frac{1}{1+2\frac{p}{\omega_2}}\right]\left[\frac{1}{1+1.2\left(\frac{p}{\omega_2}\right)+1.6\left(\frac{p}{\omega_2}\right)^2}\right]$$

This filter can be realized with two operational amplifiers, one using the high pass circuit of Figure 4a ($R_1 \rightarrow \infty$, $R_2 = 0$) and the other the low pass circuit of Figure 1c.

Narrow Band Pass Filters

A narrow band pass filter is used primarily for spectral analysis. An active filter (using operational amplifiers instead of inductors) can be designed to respond to an extremely small band of frequencies while rejecting all other components. A transformed first order low pass factor becomes:

$$\frac{1}{1+\frac{p_l}{\omega_c}} \rightarrow \frac{2B\left(\frac{p}{\omega_o}\right)}{1+2B\left(\frac{p}{\omega_o}\right)+\left(\frac{p}{\omega_o}\right)^2}$$

and a transformed second order factor becomes:

$$\frac{1}{1+2\xi_L\frac{p}{\omega_c}+\left(\frac{p}{\omega_c}\right)^2} \rightarrow \left[\frac{2B\left(\alpha\frac{p}{\omega_o}\right)}{1+2\xi\left(\alpha\frac{p}{\omega_o}\right)+\left(\alpha\frac{p}{\omega_o}\right)^2}\right]\left[\frac{2B\left(\frac{p}{\alpha\,\omega_o}\right)}{1+2\xi\left(\frac{p}{\alpha\,\omega_o}\right)+\left(\frac{p}{\alpha\,\omega_o}\right)^2}\right]$$

where for $B < \frac{1}{2}$, $\left(\frac{\omega_{CU}}{\omega_{CL}} < 2.5\right)$:

$$\xi \approx \xi_L B\left[\frac{1-\frac{B^2}{4}(1-\xi_L^2)}{1+\frac{B^2}{4}(1-\xi_L^2)}\right]$$

$$\alpha \approx \frac{1+\frac{B}{2}\sqrt{1-\xi_L^2}}{1-\frac{B}{2}\sqrt{1-\xi_L^2}}$$

Note that when B, the normalized bandwidth, is very small, the damping ratio of each band pass quadratic factor is also very small. This causes the filter to have a very slow transient response when a sudden change in spectral density of the input signal occurs. For example if the input were removed the output amplitude of a quadratic filter would decay exponentially (with envelope proportional to

$$e^{-B\omega_o t} = \exp\left\{-\left(\frac{\omega_{CU}-\omega_{CL}}{2}\right)t\right\}$$

Thus to achieve fast transient response (large damping) one must not make the pass band too narrow.

When the damping ratio, ξ, is very small the circuit of figure 2e is recommended because the noise gain is lowest, the ratio of maximum to minimum capacitance is lowest, and the impedance level at the center frequency, ω_o, is lowest. Nevertheless the choice of the amplifier and component tolerances may be critical. In many cases, an amplifier such as the P25AH or Q25AH having a high input impedance, wide bandwidth, and low current noise should be employed.

The effective bandwidth for mean square noise spectrum analysis is defined as Appendix 1:

$$b_N \equiv \int_0^\infty |A\{f\}|^2\,df$$

where the maximum value of $|A\{f\}| = 1$. This is related to the Butterworth low pass 3 db cut-off frequency as indicated in Table 3.

Table 3

Bandwidth Conversion for Butterworth Filters

Order		$\frac{\text{noise bandwidth}}{\text{3 db bandwidth}}$
Low Pass	Band Pass	
1	2	1.571
2	4	1.111
3	6	1.047
4	8	1.025
∞	∞	1.000

Example:

Design a band pass filter of the Butterworth type having a ½-octave effective bandwidth centered at 1 rad/sec (ω_o) to be used for power spectral density measurements. The filter is to have very sharp cut-off such that at $\omega = \sqrt{2}$ rad/sec, (the next highest center frequency in a comb filter) the gain is less than 0.1.

Procedure

1. Determine B

Let $\omega_{CU} - \omega_{CL} = 3$ db bandwidth.

$$B \equiv \frac{\omega_{CU} - \omega_{CL}}{2\omega_o}, \quad \omega_o \equiv \sqrt{\omega_{CU}\,\omega_{CL}} = 1 \text{ rad/sec}$$

Let $b_N = \omega_{CU}' - \omega_{CL}'$ (noise bandwidth) such that

$$\omega_o = \sqrt{\omega_{CU}' - \omega_{CL}'} = 1 \text{ rad/sec}$$

For a half octave noise bandwidth:

$$\frac{\omega_{CU}'}{\omega_{CL}'} = \sqrt{2}$$

Hence:

$$\frac{\omega_{CU}'}{\omega_o} = \frac{\omega_o}{\omega_{CL}'} = \sqrt[4]{2}$$

Call:

$$K = \frac{2\pi b_N}{\omega_{CU} - \omega_{CL}} = \frac{\text{noise bandwidth}}{\text{3db bandwidth}}$$

Values of K are presented in Table 3.

$$K = \frac{\omega_{CU}' - \omega_{CL}'}{\omega_{CU} - \omega_{CL}} = \frac{\frac{\omega_{CU}'}{\omega_o} - \frac{\omega_{CL}'}{\omega_o}}{2\,\frac{\omega_{CU} - \omega_{CL}}{2\omega_o}} = \frac{\sqrt[4]{2} - \frac{1}{\sqrt[4]{2}}}{2B}$$

Hence

$$B = \frac{\sqrt[4]{2} - \frac{1}{\sqrt[4]{2}}}{2K} = \frac{.174}{K}.$$

Using Table 3:

Order	K	B
2	1.571	.111
4	1.111	.157
6	1.047	.166

2. Determine Attenuation at $\omega = \sqrt{2}$ rad sec:

Using Table 1

Low Pass		Band Pass
$\frac{\omega_l}{\omega_c}$	$=$	$\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\frac{1}{2B}$
	$=$	$\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)\frac{1}{2B} = \frac{\sqrt{2}}{4B}$

The gain of the low pass Butterworth filter is

$$|A|^2 = \frac{1}{1 + \left(\frac{\omega_l}{\omega_c}\right)^{2n}} = \frac{1}{1 + \left(\frac{\sqrt{2}}{4B}\right)^{2n}}$$

Order ($2n$)	$\lvert A \rvert$	
2	.30	
4	.20	
6	.10	←— use this.

3. Design amplifier circuits.

The selected transfer function is: (Transformation of 3rd order Butterworth, Table 2)

$$\left[\frac{2B\frac{p}{\omega_o}}{1+2B\frac{p}{\omega_o}+\left(\frac{p}{\omega_o}\right)^2}\right]\left[\frac{2B\frac{p}{\alpha\omega_o}}{1+2\xi\frac{p}{\alpha\omega_o}+\left(\frac{p}{\alpha\omega_o}\right)^2}\right]\left[\frac{2B\frac{\alpha p}{\omega_o}}{1+2\xi\frac{\alpha P}{\omega_o}+\left(\frac{\alpha P}{\omega_o}\right)^2}\right]$$

where:

$$\omega_o = 1 \text{ rad/sec}$$

$$B = .166$$

$$\alpha \approx \frac{1 \approx \frac{B}{2}\sqrt{1-\xi_L^2}}{1 - \frac{B}{2}\sqrt{1-\xi_L^2}} = 1.155$$

$$\xi = \frac{2\xi_L B}{\alpha + \frac{1}{\alpha}} = .0822$$

$\xi_L = .5$ (damping ratio of low pass Butterworth, see Table 2).

The circuit of Figure 2c can be used for each of the quadratic factors. Choosing $\frac{C}{\xi} = 1\ \mu f$ in each case yields the circuit shown in Figure 7 whose gain vs. frequency characteristic is plotted in Figure 8.

Band Reject Filters

A band reject filter, like the band pass filter, has two cut-off frequencies. Between these two frequencies, ω_{CL} and ω_{CU}, signal components are heavily attenuated, whereas outside this band they are passed with substantial (usually unity) gain. The same terms, ω_o and B, which simplify the band pass transformation also simplify the band reject transformation.

$$\omega_o = \sqrt{\omega_{CU}\ \omega_{CL}} \quad \text{"center frequency"}$$

$$B = \frac{\omega_{CU} - \omega_{CL}}{2\omega_o} \quad \text{"normalized bandwidth"}$$

A transformed first order low pass factor becomes:

$$\frac{1}{1 + \frac{p_l}{\omega_o}} \rightarrow \frac{1 + \left(\frac{p}{\omega_o}\right)^2}{1 + 2B\left(\frac{p}{\omega_o}\right) + \left(\frac{p}{\omega_o}\right)^2}$$

and a second order low pass factor becomes:

$$\frac{1}{1 + 2\xi_L\frac{p_l}{\omega_c} + \left(\frac{p_l}{\omega_c}\right)^2} \rightarrow \left[\frac{1 + \left(\frac{p}{\omega_o}\right)^2}{1 + 2\xi\left(\alpha\frac{p}{\omega_o}\right) + \left(\alpha\frac{p}{\omega_o}\right)^2}\right]\left[\frac{1 + \left(\frac{p}{\omega_o}\right)^2}{1 + 2\xi\left(\frac{p}{\alpha\,\omega_o}\right) + \left(\frac{p}{\alpha\,\omega_o}\right)^2}\right]$$

where:

$$\xi = \frac{2\xi_L B}{\alpha + \frac{1}{\alpha}}$$

and for the very wide band case:

$$\alpha \approx 2B + \frac{1}{2B}(1 - 2\xi_L^2) \text{ for } B > 2 \left(\frac{\omega_{CU}}{\omega_{CL}} > 20\right)$$

for the very narrow band case:

$$\alpha \approx \frac{1 + \frac{B}{2}\sqrt{1 - \xi_L^2}}{1 - \frac{B}{2}\sqrt{1 - \xi_L^2}} \qquad \text{for } B < \tfrac{1}{2} \qquad \left(\frac{\omega_{CU}}{\omega_{CL}} < 2.5\right)$$

Note that the denominator is the same as that of the band pass in each case, provided that $\omega_L = \omega_c$ (See Table 1)

All-Pass Filters

An all pass circuit is not a filter in the usual sense, i.e. a device which separates desired and undesired components. Nevertheless, we apply the term filter since the all pass circuit does perform a specified operation upon an input signal in a manner similar to that of other filter circuits. The purpose of an all-pass filter is to phase shift certain frequency components while introducing no attenuation. A transfer function form which accomplishes this objective is:

$$A(p) = \frac{1 - a_1 p + a_2 p^2 - a_3 p^3 + \ldots}{1 + a_1 p + a_2 p^2 + a_3 p^3 + \ldots}$$

First-order single-amplifier circuits are shown in Figure 5a and b.

The phase shift contribution of the numerator equals that of the denominator. To achieve an all pass filter with phase lag approximately proportional to frequency, the parameters $a_1, a_2 \ldots a_n$ are chosen to be those of a Paynter low pass filter. For example the second order filter see Figure 9 is:

$$A\{p\} = \frac{1 - 3\frac{p}{\omega_o} + 4\left(\frac{p}{\omega_o}\right)^2}{1 + 3\frac{p}{\omega_o} + 4\left(\frac{p}{\omega_o}\right)^2}$$

where:

$$|A\{\omega\}| = 1$$

$$\phi\{\omega\} \approx 2\pi\frac{\omega}{\omega_o} \left(\text{(exact match at } \frac{\omega}{\omega_o} = 0, \tfrac{1}{4}, \tfrac{1}{2}\right)$$

This is an adequate model of a transport delay for some purposes, such as closed loop stability analysis. However, since the phase shift departs significantly from that of the delay at high frequencies it is important that the input contain very small high frequency components, in order that the transient response resemble that of a delay.

Higher order filters of this type can be built up from a cascade of band reject filters with a first or second order low pass, band pass, or high pass filter approximating the function:

$$\frac{1 \pm e^{-Tp}}{2}$$

The notch frequencies are those frequencies for which:

$$\frac{1 \pm e^{-j\omega T}}{2} = 0$$

Appendix 1

Noise Bandwidth:

a) Band pass Filter

The band pass filter transfer relation is derived from a low pass filter relation using the mapping relation:

$$\frac{p_l}{\omega_c} = \frac{1}{2B}\left(\frac{p}{\omega_o} + \frac{\omega_o}{p}\right)$$

such that

$$A\left\{\frac{p_l}{\omega_c}\right\} \rightarrow A\left\{\frac{1}{2B}\left(\frac{p}{\omega_o} + \frac{\omega_o}{p}\right)\right\}$$

The frequency response characteristics are related by:

$$\frac{\omega_l}{\omega_c} = \frac{1}{2B}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)$$

such that

$$\left|A\left\{\frac{\omega_l}{\omega_c}\right\}\right| \rightarrow \left|A\left\{\frac{1}{2B}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\right\}\right|$$

For example a low pass Butterworth transforms into a band pass:

$$\left|A\left\{\frac{\omega_l}{\omega_c}\right\}\right|^2 = \frac{1}{1 + \left(\frac{\omega_l}{\omega_c}\right)^{2n}}$$

$$\left|A\left\{\frac{1}{2B}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\right\}\right|^2 = \frac{1}{1 + \left(\frac{\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}}{2B}\right)^{2n}}$$

The noise bandwidth for the band pass filter is defined as:

$$b_n \equiv \int_0^\infty \left|A\left\{\frac{1}{2B}\left(\frac{f}{f_o} - \frac{f_o}{f}\right)\right\}\right|^2 df$$

$$= \frac{1}{2\pi}\int_0^\infty \left|A\left\{\frac{1}{2B}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)\right\}\right|^2 d\omega$$

Converting back to the low pass relation requires finding $d\omega$ in terms of $d\omega_l$ and ω_l.

$$\frac{\omega_l}{\omega_c} = \frac{1}{2B}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)$$

$$\left(\frac{\omega}{\omega_o}\right)^2 - 2B\left(\frac{\omega_l}{\omega_c}\right)\left(\frac{\omega}{\omega_o}\right) - 1 = 0$$

$$\frac{\omega}{\omega_o} = B\frac{\omega_l}{\omega_c} \pm \sqrt{B^2\left(\frac{\omega_l}{\omega_c}\right)^2 + 1}$$

$$\frac{d\omega}{\omega_o} = B\frac{d\omega_l}{\omega_c}\left[1 \pm \frac{B\frac{\omega_l}{\omega_c}}{\sqrt{B^2\left(\frac{\omega_l}{\omega_c}\right)^2 + 1}}\right]$$

Also when:

$$\omega \to 0 \qquad \omega_l \to -\infty$$
$$\omega \to \infty \qquad \omega_l \to \infty$$

Thus:

$$b_n = \frac{1}{2\pi}\frac{B\omega_o}{\omega_c}\int_{-\infty}^{\infty}\left|A\left\{\frac{\omega_l}{\omega_c}\right\}\right|^2 d\omega_l$$

$$\pm\frac{1}{2\pi}\frac{B^2\omega_o}{\omega_c^2}\int_{-\infty}^{\infty}\left|A\left\{\frac{\omega_l}{\omega_c}\right\}\right|^2 \frac{\omega_l\, d\omega_l}{\sqrt{B^2\left(\frac{\omega_l}{\omega_c}\right)^2+1}}$$

The first integrand:

$$\left|A\left\{\frac{\omega_l}{\omega_c}\right\}\right|^2$$

is an even function of $\frac{\omega_l}{\omega_c}$. The second integrand is an odd function. Thus the first integral is twice the integral from zero to infinity and the second integral is zero.

$$b_n = \frac{1}{2\pi}\frac{2B\omega_o}{\omega_c}\int_0^{\infty}\left|A\left\{\frac{\omega_l}{\omega_c}\right\}\right|^2 d\omega_l$$

Thus:

$$[b_n]\text{ band pass} = \frac{2B\omega_o}{\omega_c}[b_n]\text{ low pass}$$

b) Butterworth Low Pass Filter

For the Butterworth Low Pass Filter

$$\left|A\left\{\frac{\omega_l}{\omega_c}\right\}\right|^2 = \frac{1}{1+\left(\frac{\omega_l}{\omega_c}\right)^{2n}}$$

$$b_n = \frac{1}{2\pi}\int_0^{\infty}\frac{d\omega_l}{1+\left(\frac{\omega_l}{\omega_c}\right)^{2n}} = \int_0^{\infty}\frac{df_l}{1+\left(\frac{f_l}{f_c}\right)^{2n}}$$

This integral is tabulated in Standard Mathematical Tables, Student Edition, The Chemical Rubber Co., Cleveland, Ohio, 1964. p. 320, #402.

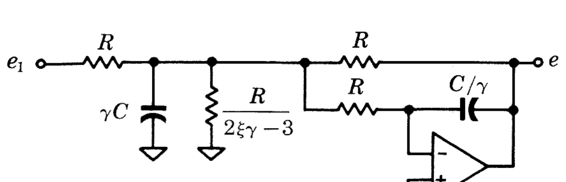
$$b_n = \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}} fc$$

Returning to the band pass filter, and employing the Butterworth characteristic

$$[b_n]\text{ band pass} = \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\,\frac{2B\omega_o}{2\pi}$$

In this case n is half the order of the filter and $\frac{2B\omega_o}{2\pi}$ is the $3db$ bandwidth in cycles per second (Hertz).

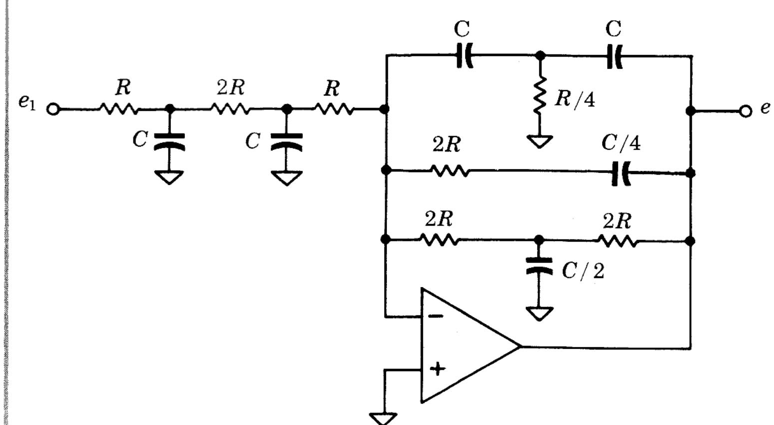
n	$\frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}} = \frac{b_n}{\frac{2B\omega_o}{2\pi}}$
1	$\frac{\pi}{2}$
2	$\frac{\pi}{2\sqrt{2}}$
3	$\frac{\pi}{3}$
$n > 3$	$\approx 1 + \frac{.41}{n^2}$



$$e = -\frac{1}{1+2\xi(RCp)+(RCp)^2}e_1$$

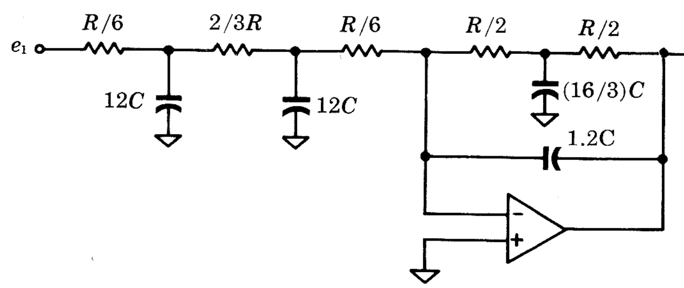
$\gamma \geq \frac{3}{2\xi}$ chosen to achieve convenient component size

Fig. 1a. Low Pass 2nd Order



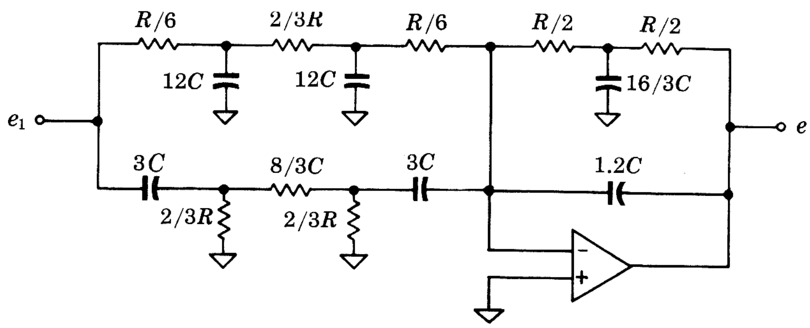
$$e = -\frac{1}{(1+RCp)[(1+RCp+(RCp)^2]}e_1$$

Fig. 1b. 3rd Order Butterworth



$$e = -\frac{1}{(1 + 2RCp)(1 + 1.2RCp + 1.6R^2C^2p^2)}\, e_1$$

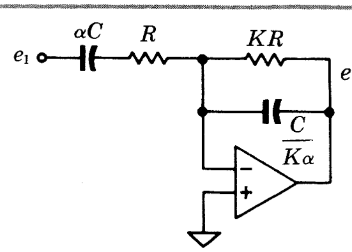
Fig. 1c. 3rd Order Paynter (Linear Phase) Filter



$$-e = -\int_{t-\pi RC}^{t} e_1 dt = \frac{1 - e^{-\pi RCp}}{\pi RCp}\, e_1 \approx \frac{1 + R^2C^2p^2}{(1 + 2RCp)(1 + 1.2RCp + 1.6R^2C^2p^2)}\, e_1$$

Fig. 1d. 3rd Order Averaging Filter

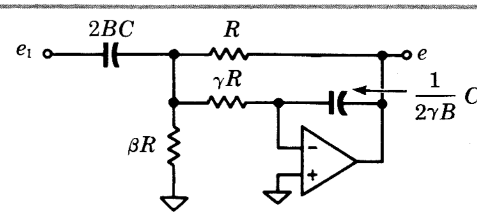
Fig. 1. Single Amplifier Low Pass Filters



$$e = -\left[\frac{\alpha RCp}{1 + \alpha RCp}\right]\left[\frac{K}{1 + \frac{RC}{\alpha} p}\right] e_1$$

$\alpha > 1$

Fig. 2a. Wide Band Pass

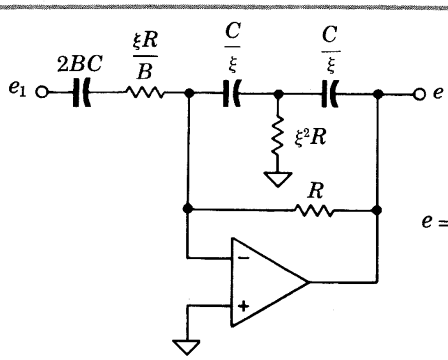


$$e = -\frac{2B(RCp)}{1 + 2\xi(RCp) + (RCp)^2}\, e_1$$

choose γ and β to satisfy:

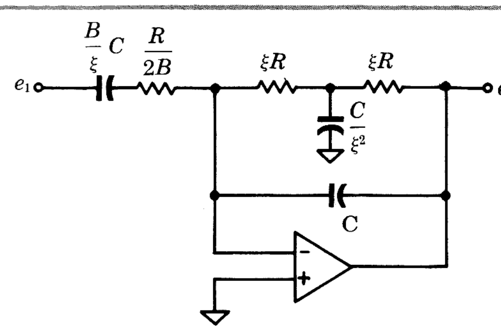
$$\frac{1}{\gamma} + \frac{1}{\beta} = 4B\xi - 1$$

Fig. 2b. Narrow Band Pass (Use for $.7 < \xi < 1$)



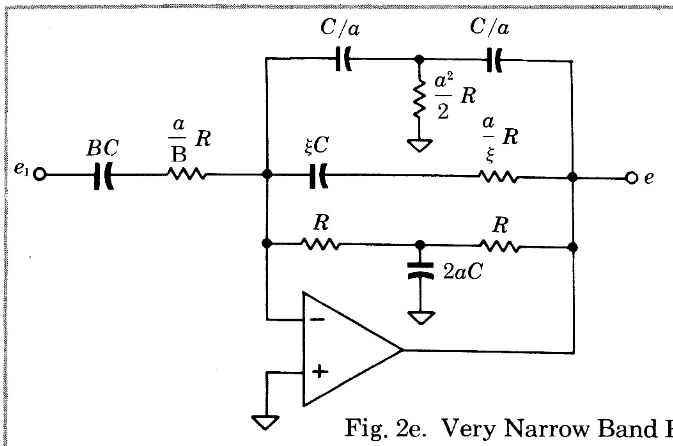
$$e = -\frac{2BRCp}{1 + 2\xi RCp + R^2C^2p^2}\, e_1$$

Fig. 2c. Narrow Band Pass (Use for $.1 < \xi < .5$)



$$e = -\frac{2BRCp}{1 + 2\xi RCp + R^2C^2p^2}\, e_1$$

Fig. 2d. Narrow Band Pass (Use for $.5 < \xi < 1$)



$$e = -\frac{2BRCp}{1 + 2\xi RCp + R^2C^2p^2} e_1$$

$$\text{Noise Gain} = \frac{1 + \left(2B + \frac{2}{a} + 2a + 2\xi\right)RCp + R^2C^2p^2}{1 + 2\xi RCp + R^2C^2p^2}$$

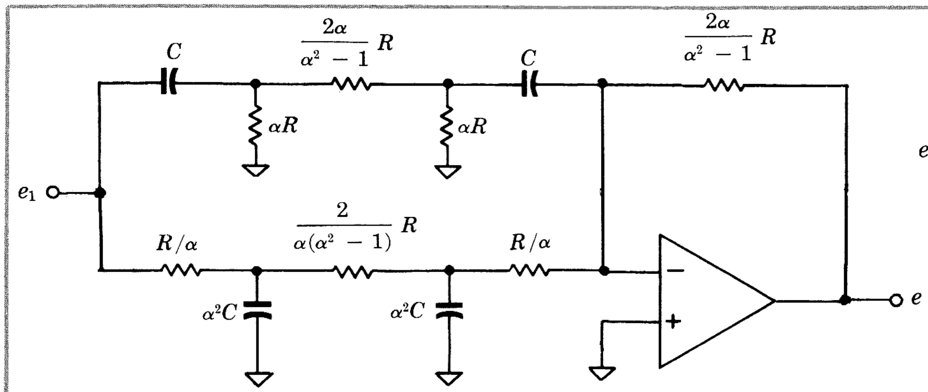
For the large capacitors all identical choose

$$a = \frac{1}{\sqrt{2}}$$

Then: $\frac{C_{max}}{C_{min}} = \frac{\sqrt{2}}{\xi}$; $\frac{C_{max}\,|Z|\text{ pass band}}{RC} = \frac{\sqrt{3}}{B}$

Fig. 2e. Very Narrow Band Pass (Use for $\xi < .35$)

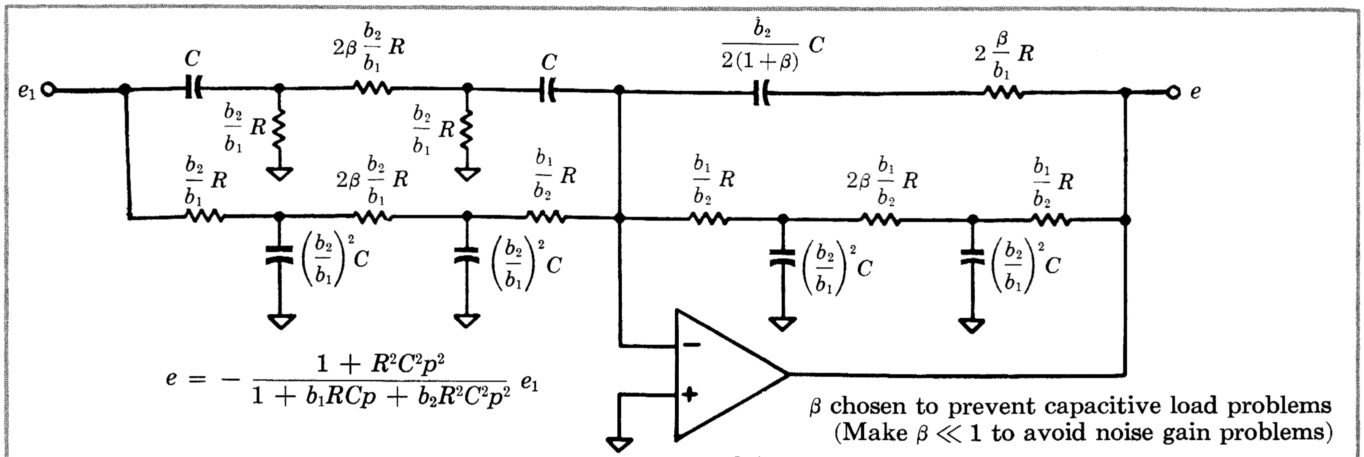
Fig. 2. Single Amplifier Band Pass Filters



$$e = -\frac{1 + R^2C^2p^2}{(1 + \alpha RCp)\left(1 + \frac{1}{\alpha} RCp\right)} e_1$$

$\alpha > 1$

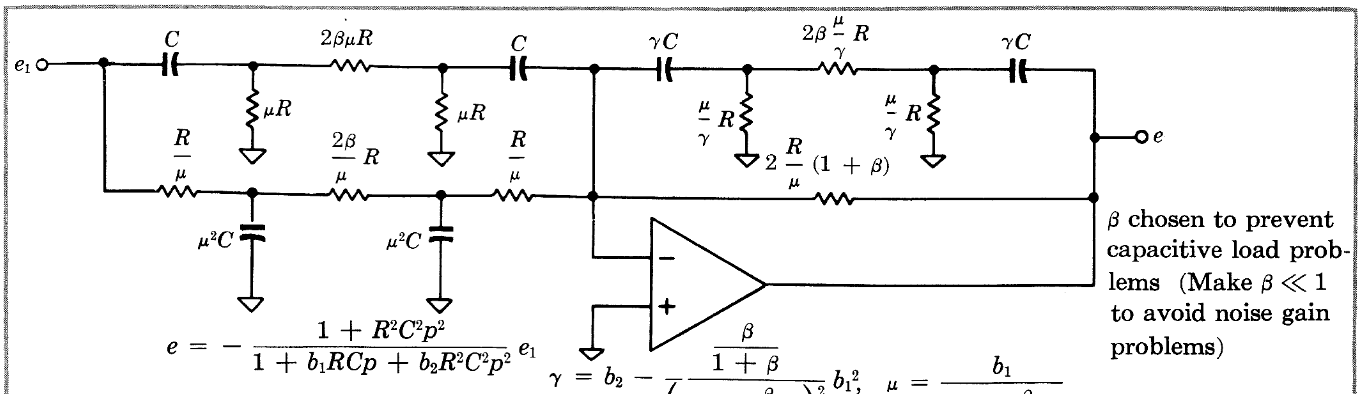
Fig. 3a. Wide Band Reject



$$e = -\frac{1 + R^2C^2p^2}{1 + b_1RCp + b_2R^2C^2p^2} e_1$$

β chosen to prevent capacitive load problems (Make $\beta \ll 1$ to avoid noise gain problems)

Fig. 3b. Narrow Band Reject ($b_1 > b_2$)



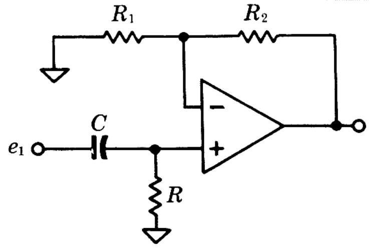
$$e = -\frac{1 + R^2C^2p^2}{1 + b_1RCp + b_2R^2C^2p^2} e_1$$

β chosen to prevent capacitive load problems (Make $\beta \ll 1$ to avoid noise gain problems)

$$\gamma = b_2 - \frac{\frac{\beta}{1+\beta}}{\left(1 + \frac{\beta}{1+\beta}\right)^2} b_1^2, \quad \mu = \frac{b_1}{1 + \frac{\beta}{1+\beta}}$$

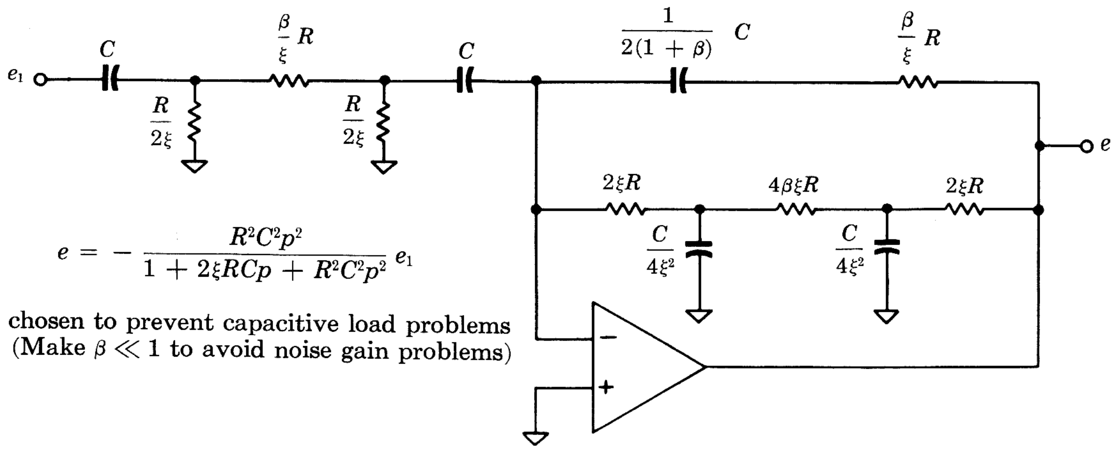
Fig. 3c. Narrow Band Reject ($b_1 < 1$)

Fig. 3. Single Amplifier Band Reject Filters



$$e = \left(1 + \frac{R_2}{R_1}\right)\left(\frac{RCp}{1 + RCp}\right) e_1$$

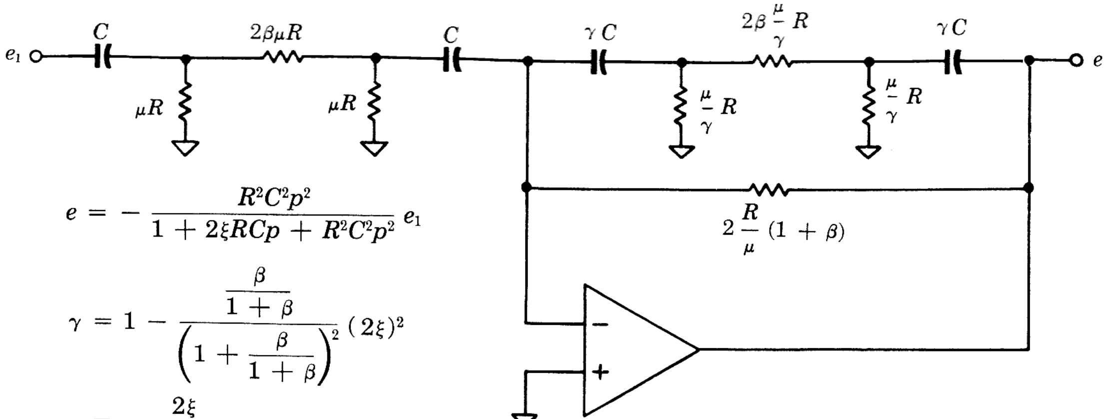
Fig. 4a. First Order



$$e = -\frac{R^2C^2p^2}{1 + 2\xi RCp + R^2C^2p^2} e_1$$

β chosen to prevent capacitive load problems (Make $\beta \ll 1$ to avoid noise gain problems)

Fig. 4b. High Pass (Use for $.5 < \xi \leq 1$)



$$e = -\frac{R^2C^2p^2}{1 + 2\xi RCp + R^2C^2p^2} e_1$$

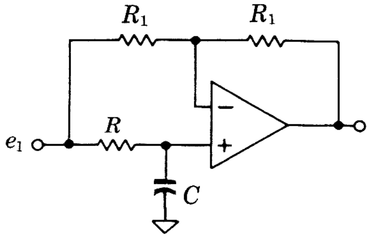
$$\gamma = 1 - \frac{\frac{\beta}{1+\beta}}{\left(1 + \frac{\beta}{1+\beta}\right)^2}(2\xi)^2$$

$$\mu = \frac{2\xi}{1 + \frac{\beta}{1+\beta}}$$

β chosen to prevent capacitive load problems (Make $\beta \ll 1$ to avoid noise gain problems)

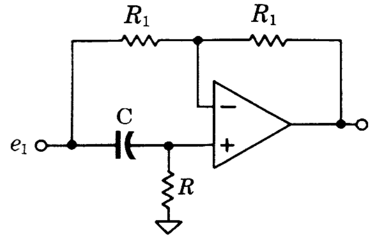
Fig. 4c. High Pass (Use for $\xi \leq .5$)

Fig. 4. Single Amplifier High Pass Filters



$$e = \frac{1 - RCp}{1 + RCp} e_1$$

Fig. 5a. First Order



$$e = -\frac{1 - RCp}{1 + RCp} e_1$$

Fig. 5b. First Order

Fig. 5. Single Amplifier All Pass Filters

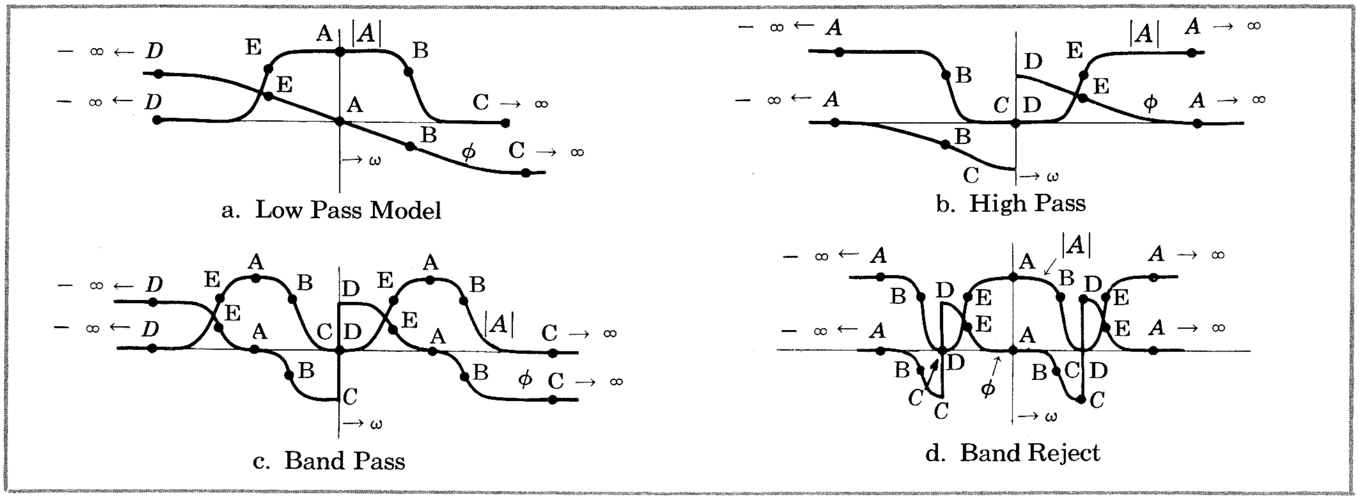


Fig. 6. Corresponding Points on Transformed
Gain and Phase Characteristics

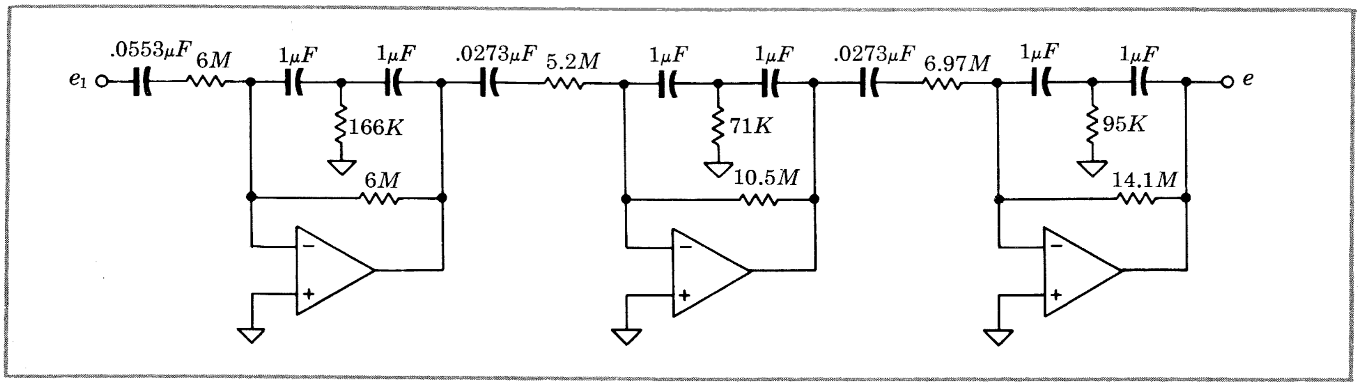


Fig. 7. Active Filter Circuit for the 6th Order Band Pass Example

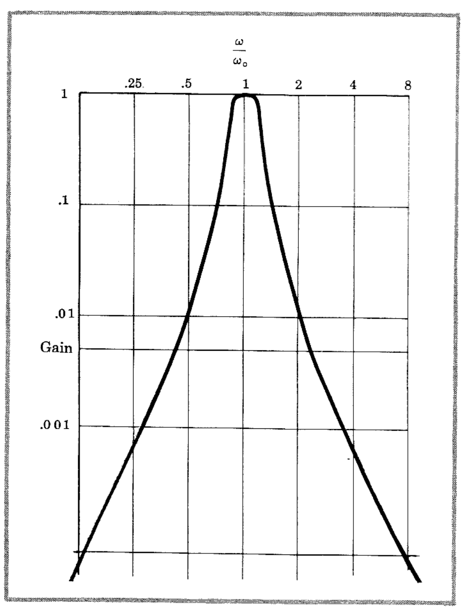
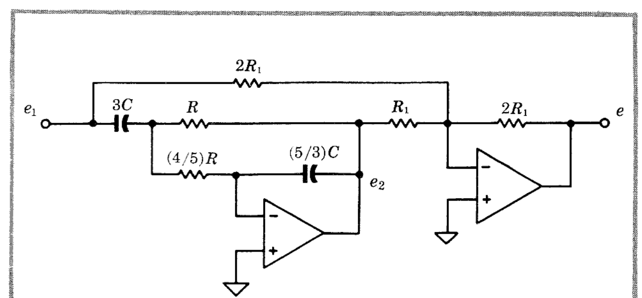


Fig. 8. Gain vs. Frequency for the 6th Order
Band Pass Example



$$e = -\frac{1 - 3RCp + 4R^2C^2p^2}{1 + 3RCp + 4R^2C^2p^2}\, e_1 \approx -\epsilon^{-2\pi RCp}\, e_1$$

$$e_2 = -\frac{3RCp}{1 + 3RCp + 4R^2C^2p^2}\, e_1 \approx -\frac{1 - \epsilon^{-2\pi RCp}}{2}\, e_1$$

Fig. 9. Second Order All Pass Filter

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