THE LIGHTNING EMPIRICIST

Advocating electronic models, at least until livelier instrumentalities emerge

Volume 13, Numbers 1 and 2 January - July 1965

PROGRESS REPORT FROM 127 TO 128

In January, 1950, we published a "PROGRESS REPORT — Tabular Summary for Customers", the predecessor of this Journal, and at that time stated:

> "General Business Condition: "We are now well into our fourth year, sound and highly active, in the black and comfortably solvent. Gross revenue for the fiscal year ended October 31, 1949 was in the six-figure bracket, while profit was maximum recorded by us to date. We are ahead of last year for the current quarter, and the outlook beyond is relatively unclouded. The bulk of our profit has been invested in the development, design, and production-engineering our series of Analog computor Components.
>
> Laboratory Moved: Our principal laboratory has been relocated in ampler and better-integrated quarters at the same address. Most of us are now to be found on the fourth floor in the same Boston building: 33% up. The mailing address is unchanged. . . ."

Fifteen years later, we are by definition older, believe ourselves therefore the wiser for having survived and prospered, employ more people, have a more diversified array of products, but little else has changed. Compare the first paragraph as it might appear today:

> "General business condition: We are now well into our nineteenth year, sound and highly active, in the black and comfortably solvent. Gross revenue for the fiscal year ended October 31, 1964 was more than halfway between the seven- and eight-figure brackets, while profit was the maximum recorded by us to date. We are ahead of last year for the current quarter, and the outlook is relatively unclouded. The bulk of our profit continues to be available for investment in the development, design, and production-engineering our series of Analog Computing Components."

Plus ça change, et plus c'est la même chose!

Now compare the second paragraph: "Since 1950, our continually pressing search for *lebensraum* has led us to occupy the fourth and sixth floors of that same building (near South Station), to abandon it in favor of one floor apiece of two buildings in the Back Bay area, and finally a basement and two floors apiece in two buildings near Back Bay Station. At long last, we have moved laboratory, offices, and production facilities to our own new plant on Route 128, near the New Haven Railroad Station." New mailing address and communication data are given elsewhere in this issue.

For the first time, we appear to be on dangerous Parkinsonian ground, moving to a plant specifically designed and constructed to fill our needs (including future expansion) and out of the city which has proved hospitable for so long. However, serious devotees of Parkinson who are sympathetic to our continued evolution will be pleased to learn that the facility has been designed with certain imperfections known only to the designer, but, which are gradually being revealed.

We are only fifteen minutes from the Back Bay by rail, 15 minutes from the airport by helicopter (our parking lot can easily be fitted with a Helipad, but a neighboring hotel already has one) and access to and from local customers along Route 128 is nearly effortless. An interesting exercise is to consider extrapolating the series of addresses and their coincidental relationship to the main line of the Railroad: If passenger service on the New Haven survives, how long will it be before we move to Providence? New York? Philadelphia, Washington?

Comparison of that early "Progress Report" with today's might reveal that there has been a shift of emphasis from problem solving alone, by repetitive computation, using the improved series of K3 high-speed dc computing components to the considerably more ramified applications of analog technology and the acceptance by the scientific community of feedback instruments as tools for measurement, functional transformations, in-line data processing, and building-block electronics. And an examination of our present and planned product lines would reveal the present ability and future intention of maintaining our pre-eminence in the design and applications of building blocks and systems for present and future devotees of feedback.

Since that early day, here are a few of the modest contributions of the Laboratory that has just moved into our new plant: the compact modular operational amplifier; the development of Selection as a concept, including as byproducts single-variable and multi-dimensional piecewise nonlinearities, multipliers, electronic "graph paper"; the application of advanced high frequency communications concepts to dc amplification; components for distortionless delay and linear synthesis; the potless patchless analog computor; the possibilities inherent in a ramified selection of operational amplifiers designed with a gamut of physical, electrical, utilitarian, and conceptual dimensional attributes.

In 1950, our unique and lovely little "Catalog and Manual", now out of print, had not yet appeared. Since then, our Publications Department has been busy indeed. Its present offerings (as of last summer) were described in some detail in an earlier issue (Vol. 12 No. 2) but they are being added to with a frequency and massiveness that cause us to shudder at the long-range extrapolations. The diversity of the fields of application is by no means a surprise; but the ready and enthusiastic use by expert practitioners in these fields of what were once considered the intimate electronic design details of Analog Computor building blocks might have been somewhat difficult to predict in 1950.

One more facet of Philbrick that is unchanged: Our interest in walking side by side with users of feedback as they explore the mysteries of the measurable Universe, unfolding old mysteries, and uncovering new mysteries.

> "Our goal is to marry operational concepts and modern electronic technology in pioneering the computing instruments you will be using tomorrow. No user is so large but what we can show him a few tricks; none is too small for our interest and affection." *G. A. Philbrick*

NEWS FROM PHILBRICK:
Brief Items

Move to new address. Since the end of February, Philbrick's new home is in our new building at
Allied Drive at Route 128 (Exit 61)
Dedham, Massachusetts 02026
Telephone: 617-329-1600
TWX: (617) 326-5754

News of Philbrick Representatives. There have been a few changes in territorial coverage in recent months, made with the express purpose of providing our customers with better service. The latest complete rep information will be found in our "Directory", available upon request. In brief, here are the most significant changes:

Texas (except El Paso), Oklahoma, Western Louisiana are covered by Applied Science Associates, from their offices in Dallas and Houston. Telephone numbers: Dallas, 214-526-8316; Houston, 713-781-1441. Jack Blacketer and Oliver Kollock cover Dallas and points North and West; Herb Bloschies covers Houston and points South and East.

New Orleans, Louisiana is now covered by W. A. Brown and Associates, Inc. from the New Orleans office, telephone: 504-242-5575. Jim McCord is the man to talk with.

Kentucky, except for Boone, Kenton, and Campbell counties, is now covered by Hugh Marsland and Co., from their Indianapolis office, telephone: 317 FLeetwood 6-4249. Bud Riedy is the man on the spot. Boone, Kenton, and Campbell counties, across the Ohio River from Cincinnati, are covered by Herm Martin, of M. P. Odell Co., Dayton, Ohio, telephone: 513-298-9964.

Kansas City, Missouri (and the entire state of Kansas) are now covered by Hugh Marsland and Company. Bernie Gosselin, at Kansas City, telephone: 913-381-2122, mans the bastions.

The Montreal office of Ahearn and Soper in Canada has a new address and telephone number: 3300 Cavendish Blvd., Suite 220, Montreal 28, telephone: 514-482-9750. It is manned by Doug McCormick. Business as (or better than) usual in Toronto, served by Walter Borlase and Joe Paul — telephone: 416 RUssell 9-4325.

Augmented and Special Service. Three of our representatives have established allied engineering operations to assist local customers who may require services above and beyond the normally copious free consulting available from the Philbrick family, particularly in regard to construction of special operational structures based on Philbrick computing modules. These operations have their own special competence, yet work in close conjunction with our Applications Staff.

On the West Coast, the engineers at Tech-Stok are available to discuss your engineering problems in depth — and do something concrete about them. Phone 213 WEbster 7-0780 in Los Angeles, 415 DAvenport 6-9800 in the Bay area, 714 ACademy 2-1121 in San Diego, and 602-265-3629 in Phoenix, Arizona.

In the New York - Philadelphia area, Bob Crane of Electronic Gear, Inc. is on tap to help solve your problems with specially engineered combinations of Philbrick modules and appropriate items of electronic hardware. Phone 516 - LOcust 1-8051 on Long Island, 215-277-0559 in the Philadelphia area.

In Wakefield, Mass., Bill Baker and his colleagues at Teal Engineering can help reduce devilish dilemmas to friable form in sturdy structures employing Philbrick products. Phone 617-245-9060 in Massachusetts, 203-233-5503 in Connecticut.

Philbrick now publicly-held. After many years of private ownership, and following a successful offering of our common shares, the public — including many of our own employees — now participates in the ownership of Philbrick Researches.

NEW PRODUCTS

Models P25A and PP25A Differential Operational Amplifiers

Plug-in Model P25A and its wire-in counterpart, Model PP25A, combine high input impedance, low offset current, moderately low offset voltage and wide bandwidth in versatile operational units useful for integration, sample-and-hold, track-and-hold, precise isolation and amplification, non-loading differential voltage crossing detection, and low-level logarithmic operations. P25A is directly interchangeable with amplifiers ranging from P35A to P85A and may be used in crucial applications where the unit you now have has insufficient input impedance or excessive offset current.

Model P25A, a premium unit, uses hermetically-sealed silicon transistors throughout, and its input circuit uses specially selected and matched field effect transistors. At low frequencies, at 25°C, input impedance is in excess of 10^{11} ohms, both differentially, and from either input to common. Input offset current at either input is less than 150 picoamperes, and remains less than 1 nanoampere to temperatures as high as 50°C. In P25A, an accessible internally-mounted variable resistor is provided to allow adjustment of offset voltage to zero, and offset voltage changes less than 6 millivolts between −25°C and +85°C. DC gain is 20,000 minimum, gain-bandwidth is greater than 1.0 Mcps, and full output of ±11 volts at 2 milliamperes is available to at least 10 kcps. The amplifiers require ±15 volts dc at ±5.5 milliamperes quiescent from either supply, plus external load current. Size and weight are similar to corresponding quantities for most other P- and PP-size amplifiers, and price is substantially less than that of Model P2A.

Further details are available upon request from Philbrick's Applications Department.

Models P65Q and PP65Q

These economical units employ all hermetically-sealed silicon semiconductors, have extremely low quiescent power drain, and were specifically designed for battery-powered applications. In many ways similar to Models P65A and PP65A, they will serve with distinction in field computing applications, as amplifiers, adders, subtractors, comparators, transducer amplifiers, current-to-voltage and voltage-to-current converters and for a host of nonlinear applications.

Total power drain is less than 30 milliwatts at ±15 volts quiescently, and full output of ±11 V at ±0.55 ma is available into a load of 20k. When used inside the loop with Model P66A Booster Follower, full output of ±10 volts at ±100 milliamperes (i.e., 1 watt) is available, the total quiescent drain being less than 3.5 ma for both units.

At 25°C, Models P65Q and PP65Q have open-loop dc gain of 20,000 or more, gain-bandwidth greater than 1.3 Mcps, full output at frequencies as great as 5 kcps, dc differential input impedance greater than 150k, and more than 20 megohms of common mode input impedance. Voltage offset adjustment of P65Q is externally effected via an accessible variable resistor; PP65Q is trimmed by an external resistor. Offset variation of P65Q is typically less than 50 microvolts per day, and absolutely less than 6 millivolts from −25°C to +85°C, referred to the input. Current offset is less than 10 nanoamperes per day at constant temperature at either input, and is typically less than 125 nanoamperes from 25°C to 85°C or 250 nanoamperes from 25°C to −25°C.

THE LIGHTNING EMPIRICIST

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NEW APPROACHES TO THE DESIGN OF ACTIVE FILTERS

Peter D. Hansen, ScD
Vice President, Philbrick Researches

Summary

Operational amplifiers can greatly simplify the design of high performance signal filters because they eliminate the need for inductors and for impedance matching. Furthermore, use of active filters can result in reduction of weight, size, and cost. Filters designed to satisfy sophisticated mathematical criteria can be realized without resort to "equalization" or trimming.

In this issue we discuss the design of operational amplifier and analog computor circuits suitable for use as low pass filters. We also discuss the commonly used mathematically designed filters, i.e. Butterworth, Chebyshev, and Bessel. In addition, we present two new types of theoretical filters, the Paynter and the Averaging filters. Design data necessary for realizing these theoretical filters with amplifier circuits is provided.

In the next issue we shall discuss the design of band pass, band reject, high pass and all pass active filter circuits.

Review of Terminology

In the context of this discussion, a filter is a circuit which performs a designated dynamic operation upon a single input signal. Usually the purpose of such a circuit is to provide an output which has the same Fourier composition as the input in a certain frequency band and no components outside this band.

The output, y, of an nth order linear filter can be found in terms of the input, x, by solving an nth order linear differential equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y$$

$$= c_n \frac{d^n x}{dt^n} + c_{n-1} \frac{dx^{n-1}}{dt^{n-1}} + \cdots + c_1 \frac{dx}{dt} + c_0 y$$

where a_i and c_i are usually time invariant parameters of the filter. These or related parameters must be selected by the filter designer. Note that we are not limiting this review to low pass filter circuits. A more convenient form from the point of view of filter synthesis results when the Heaviside Operator, $p = \frac{d}{dt}$, is employed.

$$\frac{y}{x} = A(p) = \frac{c_0 + c_1 p + \cdots + c_n p^n}{a_0 + a_1 p + \cdots + a_n p^n}$$

$A(p)$ is called the transfer function, or more properly the transfer operator.

When x is a sine wave function of time:

$$x = X \sin \omega t$$

$$px = \frac{dx}{dt} = X\omega \cos \omega t$$

$$p^2 x = \frac{d^2 x}{dt^2} = -X\omega^2 \sin \omega t = -\omega^2 x$$

Thus:

$$p^2 = -\omega^2$$
$$p = j\omega, \quad j = \pm\sqrt{-1}$$

The frequency response can be found using this relationship.

$$\frac{y(\omega)}{x(\omega)} = A(\omega) = \frac{[c_0 - c_2\omega^2 + c_4\omega^4 \cdots] + j[c_1\omega - c_3\omega^3 + c_5\omega^5 \cdots]}{[a_0 - a_2\omega^2 + a_4\omega^4 \cdots] + j[a_1\omega - a_3\omega^3 + a_5\omega^5 \cdots]}$$

The gain is given by:

$$|A(\omega)|^2 = \frac{[c_0 - c_2\omega^2 + c_4\omega^4 \cdots]^2 + [c_1\omega - c_3\omega^3 + c_5\omega^5 \cdots]^2}{[a_0 - a_2\omega^2 + a_4\omega^4 \cdots]^2 + [a_1\omega - a_3\omega^3 + a_5\omega^5 \cdots]^2}$$

and the phase lag by:

$$\phi = \tan^{-1}\left[\frac{a_1\omega - a_3\omega^3 + a_5\omega^5 \cdots}{a_0 - a_2\omega^2 + a_4\omega^4 \cdots}\right] - \tan^{-1}\left[\frac{c_1\omega - c_3\omega^3 + c_5\omega^5 \cdots}{c_0 - c_2\omega^2 + c_4\omega^4 \cdots}\right]$$

At very low frequencies a truncated Taylor series is adequate to approximate the gain:

$$\lim_{\omega \to 0} |A(\omega)| \approx \left|\frac{c_0}{a_0}\right| \left[1 - \frac{T_s^2 \omega^2}{2} + [\quad]\omega^4 \cdots\right]$$

$$\text{where: } T_s^2 = \left[\left(\frac{a_1}{a_0}\right)^2 - 2\frac{a_2}{a_0}\right] - \left[\left(\frac{c_1}{c_0}\right)^2 - 2\frac{c_2}{c_0}\right]$$

The parameter, T_s, which we call the dispersion time, has particular significance for the Gaussian Types of filters discussed in a later section. At very low frequencies the phase lag is approximately linear with frequency:

$$\lim_{\omega \to 0} (\phi) \approx \left(\frac{a_1}{a_0} - \frac{c_1}{c_0}\right)\omega - [\quad]\omega^3 \cdots$$

At very high frequencies the gain may be approximated by:

$$\lim_{\omega \to \infty} |A(\omega)| = \left|\frac{c_m}{a_n}\right| \omega^{m-n}$$

where m is the largest exponent of the numerator having a non-zero coefficient. Also at very high frequencies the phase lag may be approximated by:

$$\lim_{\omega \to \infty} (\phi) = (n - m)\frac{\pi}{2} - \left(\frac{a_{n-1}}{a_n} - \frac{c_{m-1}}{c_m}\right)\frac{1}{\omega}$$

As an example let us consider the quadratic (second order) transfer function:

$$A(p) = \frac{1}{1 + \frac{2\xi}{\omega_N} p + \frac{1}{\omega_N^2} p^2}$$

where $\xi \equiv$ damping ratio

$\omega_n \equiv$ natural frequency

$$|A(\omega)|^2 = \frac{1}{\left[1 - \left(\frac{\omega}{\omega_N}\right)^2\right]^2 + 4\xi^2 \left(\frac{\omega}{\omega_N}\right)^2}$$

$$\lim_{\omega \to 0} |A(\omega)| \approx 1 - \frac{T_s^2 \omega^2}{2} \approx \frac{1}{1 + \frac{T_s^2 \omega^2}{2}}$$

$$T_s^2 = \frac{2}{\omega_N^2}[2\xi^2 - 1]$$

$$\lim_{\omega \to \infty} |A(\omega)| = \left(\frac{\omega_N}{\omega}\right)^2$$

$$\phi = \tan^{-1} \frac{2\xi \frac{\omega}{\omega_N}}{1 - \left(\frac{\omega}{\omega_N}\right)^2}$$

$$\lim_{\omega \to 0} \phi = 2\xi \frac{\omega}{\omega_N}$$

$$\lim_{\omega \to \infty} \phi = \pi - 2\xi \left(\frac{\omega_N}{\omega}\right)$$

when $\omega = \omega_N$

$$|A(\omega_N)| = \frac{1}{2\xi}, \quad \phi = \frac{\pi}{2}$$

Also it can be shown that the greatest gain sensitivity to frequency occurs at $\frac{\omega}{\omega_N} \approx \frac{1 \pm \xi/2}{1 \mp \xi/2}$ where:

$$\frac{d\,ln\,|A|}{d\,ln\left(\frac{\omega}{\omega_n}\right)} \approx \frac{1}{2\xi} \quad \text{and} \quad \frac{|A|}{|A|_{\max}} = \frac{1}{\sqrt{2}}, \quad |A|_{\max} = \frac{1}{2\xi}$$

provided $\xi \ll 1$. This result is helpful in assessing the importance of component tolerance on the filter gain characteristic.

For our purposes it is convenient to consider the Heaviside Operator to be a complex variable:

$$p = \sigma + j\omega$$

such that the sine wave input and output becomes a special case, $\sigma = 0$. Those values of the complex variable, p, which cause the denominator of the transfer function to be zero are called poles. Those values of p which cause the numerator to be zero are called zeros. The denominator and numerator can be expressed in terms of these roots: poles p_i, and zeros z_i, respectively.

$$\frac{y}{x} = A(p) = \frac{c_n(p - z_1)(p - z_2)\cdots(p - z_n)}{a_n(p - p_1)(p - p_2)\cdots(p - p_n)}$$

$$= \frac{c_0\left(1 - \frac{p}{z_1}\right)\left(1 - \frac{p}{z_2}\right)\cdots\left(1 - \frac{p}{z_n}\right)}{a_0\left(1 - \frac{p}{p_1}\right)\left(1 - \frac{p}{p_2}\right)\cdots\left(1 - \frac{p}{p_n}\right)}$$

Stability requires that $y(t)$ remain within finite bounds for any finite $x(t)$. Also we require that, if $x(t)$ were constant, $y(t)$ must relax to a constant value. Physical realizability requires that $y(t)$ not respond before a disturbance in $x(t)$ occurs. To achieve stability and physical realizability it is necessary and sufficient that the real parts of all poles be negative, that the gain at zero frequency, $\left|\frac{c_o}{a_o}\right|$, be finite, and that the order of the numerator not exceed that of the denominator.

It can be shown that all complex poles and zeros must occur in conjugate pairs ($\sigma \pm j\omega$) provided the numerator and denominator polynomials have all real coefficients. Hence the numerator and denominator can be factored into first and second order terms each having real parameters:

$$\text{First order:} \quad \left(1 - \frac{p}{\sigma}\right) = 1 + \frac{p}{\omega_N}$$

$$\text{Second order:} \quad \left(1 - \frac{p}{\sigma + j\omega}\right)\left(1 - \frac{p}{\sigma - j\omega}\right)$$

$$= \left(1 + \frac{2\xi}{\omega_N} p + \frac{1}{\omega_N^2} p^2\right)$$

where:

$\omega_N = +\sqrt{\sigma^2 + \omega^2}$ natural frequency

$\xi = -\frac{\sigma}{\omega_N}$ damping ratio $\left(Q = \frac{1}{2\xi}\right)$

The order, n, of the filter is defined as the largest denominator exponent of p whose coefficient is not zero. Any even order filter transfer function may be realized as the cascade of 2nd order or quadratic transfer functions.

$$A(p) = \frac{c_0}{a_0} \prod_{i=1}^{n/2} \left[\frac{1 + c_{1_i}p + c_{2_i}p^2}{1 + a_{1_i}p + a_{2_i}p^2}\right]$$

Odd order filter transfer functions may be put in the form:

$$A(p) = \frac{c_0}{a_0}\left[\frac{1 + c_{10}p}{1 + a_{10}p}\right] \prod_{i=1}^{(n-1)/2} \left[\frac{1 + c_{1_i}p + c_{2_i}p^2}{1 + a_{1_i}p + a_{2_i}p^2}\right]$$

With some lack of rigor, the Laplace Transform variable, s, can be used interchangeably with the Heaviside Operator, p. We prefer the operational method to the transform method because there is no need to specify values for initial ($t = 0$) conditions. Furthermore, we have no need to make inverse transformations since the analog computor is far more efficient for studying time domain behavior.

Analog Computor Filter Circuits

Dr. Earl Wood and Mr. Ralph Sturm are successfully using the Mayo Clinic's (Rochester, Minnesota) SK5 computer programmed as an eighth order low-pass Butterworth filter to separate mechanically induced noise and high-frequency non-specific effects from physiological signals. These filters are particularly useful in experiments being carried out on a human centrifuge concerning the cardiopulmonary effects of accelerations such as encountered during the launch and re-entry phases of space flight. This investigation is supported in part by a research grant from the National Aeronautics and Space Administration. The computer circuit provides no more than 2 per cent attenuation in the useful pass band and 98 per cent attenuation in the stop band. The transition is merely an octave wide. Typical input and output waveforms are shown in Figure 1.

Most even order high performance theoretical filters have all complex poles. Odd order filters generally have only one real pole. As has been shown, an even order filter whose roots, or quadratic factors are known can be synthesized as a cascade of second order filters. Figure 2b shows an SK5 computor block diagram for a second order filter. Two SK5-U's, each

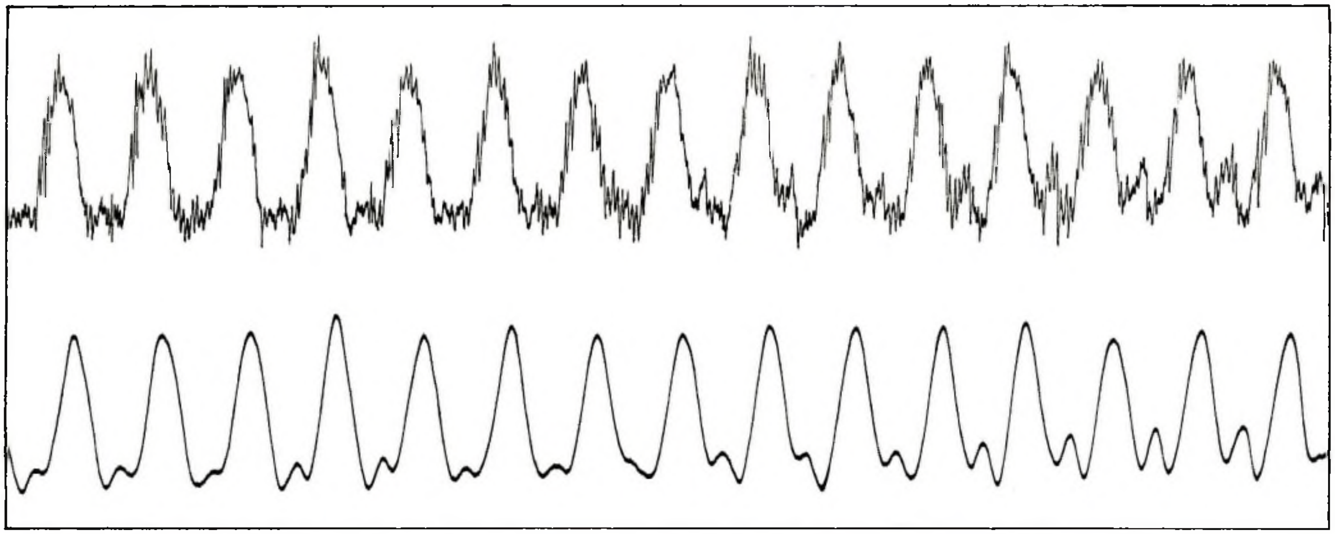


Fig. 1. Recording of pressure in the pulmonary artery during an exposure to an acceleration of 6G on a centrifuge (radius of rotation: 14½ feet).

The pressure was recorded via a catheter, 80 cm. in length, with an inside diameter of approximately 1 mm. The catheter, which was filled with fluid (sterile Ringer's solution) was connected to an unbonded strain-gauge manometer (Statham strain-gauge). This recording was obtained during a study of the effects of acceleration on the cardiovascular and pulmonary systems. This level of acceleration is similar to that experienced by astronauts during the launch and re-entry phases of space flights.

Upper trace: Unfiltered recording used as input to 8th order Butterworth filter.

Lower trace: Filtered recording, i.e., the simultaneous output from the Butterworth filter. Note the time lag introduced by the filter.

The cut-off frequency of the filter (f_c) as 11.94 cycles per second.

Paper speed of recorder: 75 mm./second.

consisting of two operational amplifier circuits, are required per quadratic stage. Independent selection of natural frequency and damping ratio (or "Q") are achieved using switch-set decade coefficients. Frequencies ranging from 10^{-1} to 10^4 rad/sec are practical. (Lower frequencies can also be handled with some difficulty.) Damping ratios ranging from .001 to 1.00 can be realized.

Although a general purpose computor can be used even when denominator factors are not known, scaling so that internal maximum voltages are not exceeded is more difficult and generally involves some trial and error. As a first step it is convenient to normalize the transfer function so that the coefficients of the highest and lowest order denominator terms are unity or nearly unity ($a_0 = 1$, $a_n \approx 1$) thus achieving the following form:

$$\frac{y}{x} = \frac{1 + c_1\left(\frac{p}{\omega_c}\right) + c_2\left(\frac{p}{\omega_c}\right)^2 + \cdots + c_n\left(\frac{p}{\omega_c}\right)^n}{1 + a_1\left(\frac{p}{\omega_c}\right) + a_2\left(\frac{p}{\omega_c}\right)^2 + \cdots + a_n\left(\frac{p}{\omega_c}\right)^n}$$

This method of preliminary scaling is successful when the natural frequencies of all the roots are of the same order of magnitude as is the case with most theoretical filters. There are many possible block diagrams usable for solving this equation. Two are shown in Figures 2c and d. The format of Figure 2c was used in generating the step responses shown in Figure 9. The SK5 computor is particularly well suited for experimentation

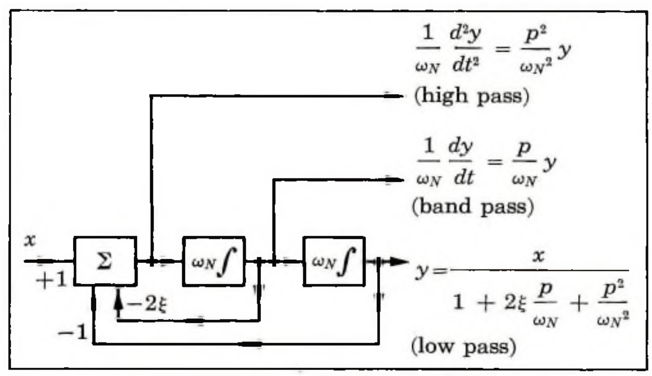


Fig. 2a. Mathematical Block Diagram for a Second-Order Filter.

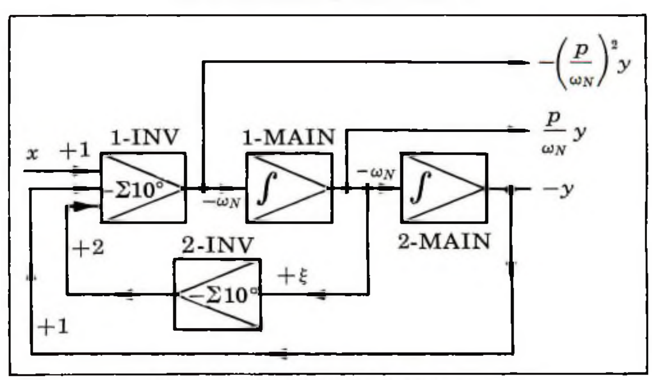


Fig. 2b. SK5-U Block Diagram for a Second-Order Filter.

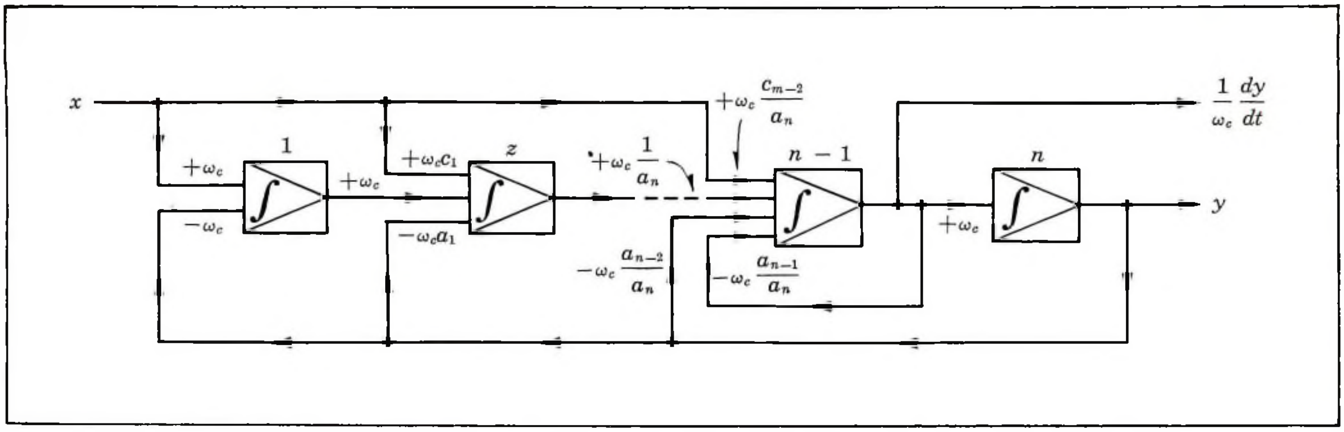


Fig. 2c. SK5-U Block Diagram for an nth-order filter when the order of the denominator is two or more than the order of the numerator.

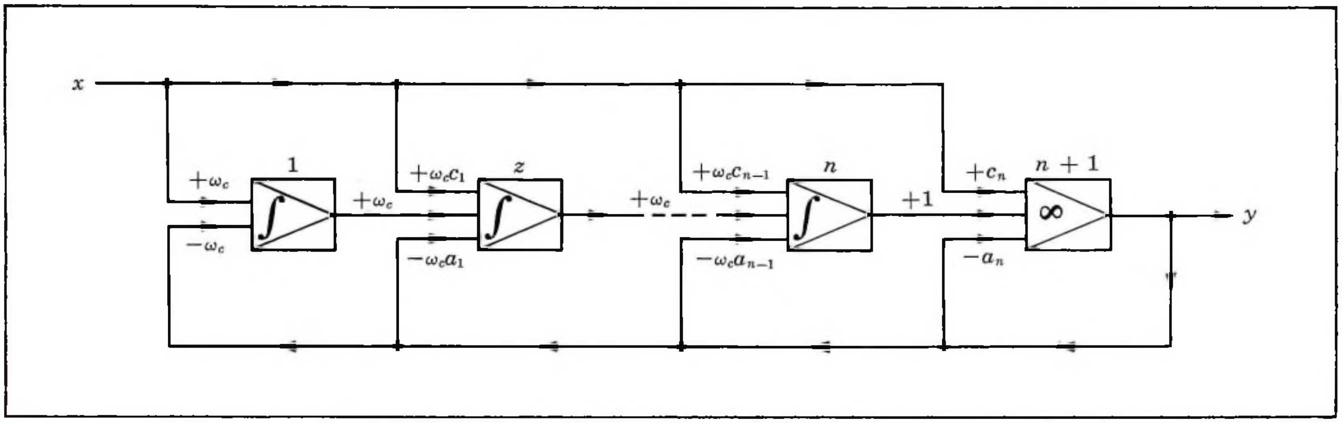


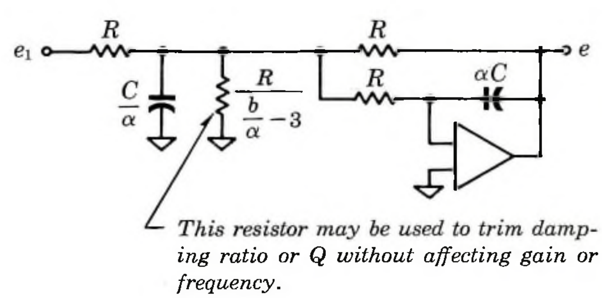
Fig. 2d. SK5-U Block Diagram for an nth-order filter when the order of the numerator is less than or equal to that of the denominator. This circuit can be achieved with n SK5-U's instead of the $n + 1$ shown when $n \geqslant 3$ if the main and inverter amplifiers are used separately.

with various filter types and cutoff frequencies because parameters are controlled by switch set decade coefficients which are easily changed.

Special Amplifier Low Pass Filter Circuits

When a filter is to be used for an extensive period of time a special-purpose circuit which is much less easily changed but less expensive may be more appropriate. Provided the roots or quadratic factors of the denominator are known, an even order filter can be synthesized with a cascade of single amplifier second order filters such as the low pass section shown in Figure 3a. Also indicated in this figure is an estimate of the maximum $\left(Q \approx \frac{1}{b}\right)$ which depends upon the amplifier gain, $\omega_H RC$, at the filter resonant frequency. Other factors affecting the choice of amplifier and impedance level for a particular application include the need to employ impedances high enough so that the source is not loaded (a low impedance voltage source is assumed) and impedances low enough so that the amplifier's error or noise current does not introduce significant voltage error at the output.

These requirements, together with the constraint imposed by practical component sizes, limit application to the frequency range, 10^{-2} to 10^5 cps.



$$e = -\frac{1}{1 + bRCp + R^2C^2p^2}\, e_1$$

$\alpha \leq \frac{b}{3}$ (Chosen to achieve convenient capacitor values)

$b^2 > \frac{3}{\omega_H RC}$ (Amplifier Gain-Bandwidth Limitation)

Input Impedance $\approx R$

Fig. 3a. Low Pass Second Order Circuit

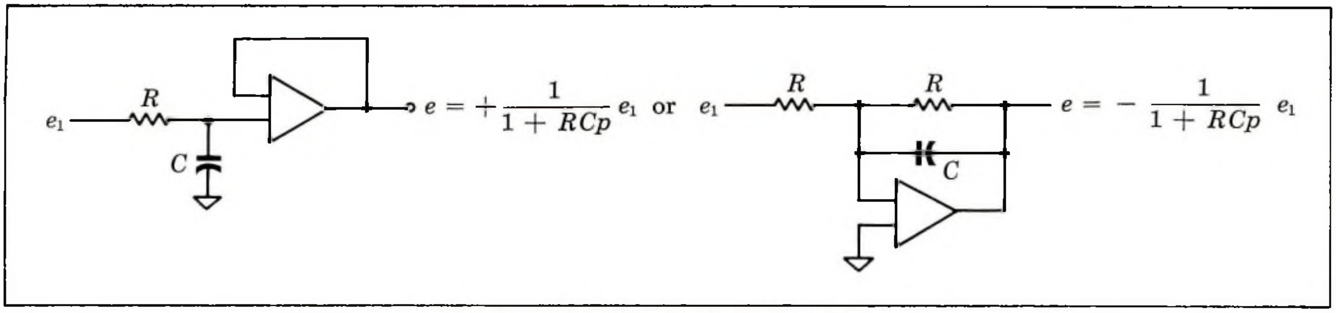


Fig. 3b. Low Pass First Order Circuit

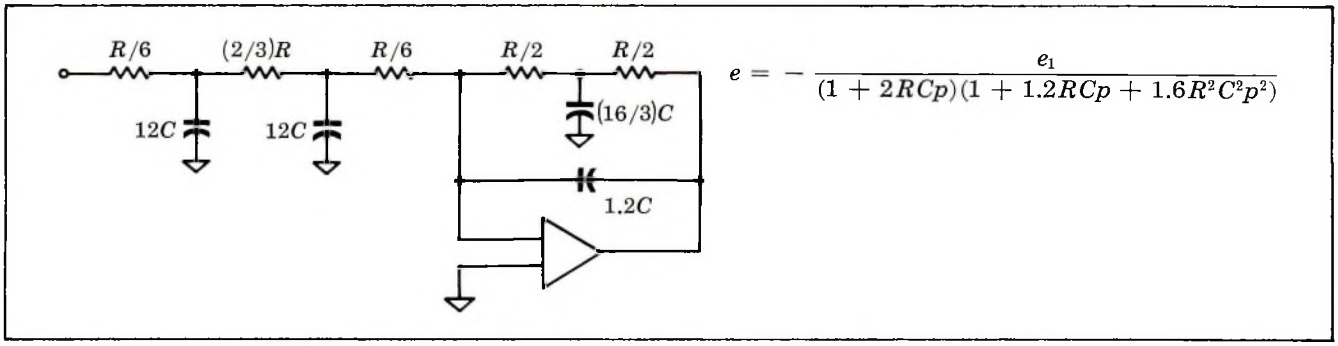


Fig. 3c. Third Order Paynter Filter Circuit

Generally low pass filters are not overly sensitive to component tolerances. However, as has been mentioned, for frequencies in the neighborhood of the natural frequency of a very lightly damped quadratic, there can be a rather large change in gain resulting from a shift in the natural frequency. Even so, this may not be of great significance, since at such frequencies the gain is usually attenuated. For example, consider the eighth order Butterworth filter where the most underdamped quadratic factor has a damping ratio of .195. Suppose that the component tolerances cause the natural frequency of this factor to be in error by 1 per cent. At

$$\frac{\omega}{\omega_N} = \frac{1-\frac{\xi}{2}}{1+\frac{\xi}{2}} = .822$$

the resulting gain error is maximum and will be:

$$\delta \ln |A| = \frac{1}{2\xi}\,\delta \ln \omega_N = \frac{.01}{.39} = .0256$$

At this frequency the ideal gain would be .975. Hence, the possible 2.56 per cent error could cause a negligible peaking of the gain characteristic which in the ideal case is optimally flat.

Figure 4 shows an additional filter circuit which may lead to a more economical design if high performance is not required. The transfer function of this filter has all real poles and therefore it does not possess as flat a band pass nor as sharp a cut-off as the more exotic filters which are the primary subject of interest in this discussion.

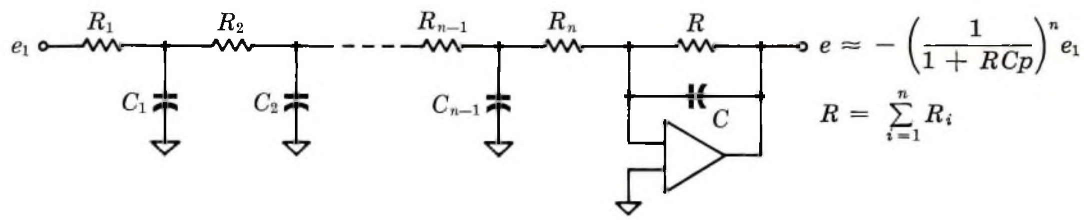
Theoretical Low Pass Filters

Low pass filters are useful for stripping analog signals of their high frequency components. This is of particular importance preceding a periodic sampling operation since it can be shown that signal Fourier components above half the sampling frequency become indistinguishable from those less than half the sampling frequency and thus contribute uncorrectable error. A sophisticated filter design is required whenever the input Fourier components which are to be transmitted are not far removed in frequency from those which are to be rejected. In general, perfect separation of desired signal components from those not desired is physically unrealizable. Thus, depending upon the application, certain aspects of filter performance must be sacrificed in order to optimize others.

Theoretical low pass filters may be separated into three categories depending upon the aspect of filter performance which is optimized, frequency response, transient response, or statistical performance. In the first, the desired characteristic is specified in terms of the frequency response. For example, an ideal filter of this type has unity gain in the pass band and zero gain in the stop band with the transition occurring in an infinitesimal band.

In the second type, the desired characteristic is specified in terms of the step or impulse response. Examples of this type are the Gaussian (Linear Phase) and Averaging filters. These will be discussed in detail in the succeeding sections.

The third type is designed to maximize the signal to noise ratio in a statistical sense by taking into account the precise spectral densities or correlation functions of the signal and noise components of the input such as is done in the application of the Wiener-Hopf equation. This type will not be covered in this discussion because the resulting filter transfer function typically has a very simple form (and the theory is relatively complicated). For example, the optimum (Wiener-Hopf) filter for separating an aperiodic square



To achieve nearly equal time constants make:

n even:

$R_1 = R_n \ll R_2 = R_{n-2} \ll \cdots \ll R_{n/2} = R_{(n/2)+1}$

$C_1 = C_{n-1} \gg C_2 = C_{n-2} \cdots \gg C_{n/2}$

$R_1 C_1 \approx \cdots \approx \frac{1}{2} R_{n/2} C_{n/2} \approx RC$

n odd:

$R_1 = R_n \ll R_2 = R_{n-2} \ll \cdots \ll R_{(n+1)/2}$

$C_1 = C_{n-1} \gg C_2 = C_{n-2} \gg \cdots \gg C_{(n-1)/2} = C_{(n+1)/2}$

$R_1 C_1 \approx R_2 C_2 \approx \cdots \approx R_{(n-1)/2} C_{(n-1)/2} \approx RC$

Fig. 4. n th Order Low Pass Having Real Poles

wave (telegraph) signal, whose probability of switching is described by a Poisson distribution $\left(\phi_{SS} = \frac{1 - a^2}{1 - p^2}\right)$, from a white noise background $(\phi_{NN} = a^2)$ is a first order lag $\left(\frac{1 - a}{1 + ap}\right)$.* This result is also obtained for other commonly encountered signal waveforms. However the method requires that the relative noise at the input (a) be specified. Often this is not easily estimated.

Frequency Response Designs

The ideal gain vs. frequency characteristic of a selective low pass filter is shown in Figure 5. Both the well known Butterworth and Chebyshev types of filters are finite order approximations of this characteristic.

The Butterworth design philosophy has much in common with that implied when a truncated Taylor series is used to approximate a transcendental function. the nth order Butterworth filter matches the ideal gain (unity) and its first n-1 derivatives (all zero) with respect to frequency at zero frequency. Thus the resulting gain vs. frequency characteristic is said to be optimally flat at zero frequency. At the cut-off frequency, ω_c, which nominally separates the pass band, from the stop band, the gain is .707 (or −3db).

$$|A|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

The gain vs. frequency characteristic for even order Butterworth filters is shown in Figure 5b. As the order increases, the ideal characteristic is more closely approximated.

Although the Butterworth design philosophy takes no account of phase, the phase shift may be of significance particularly as it provides a measure of transient distortion. Figure 5d is a plot of the departure from phase shift proportional to frequency at the cut-off frequency as a function of the order, n. This departure increases with increasing n as does the overshoot in the step response. A computer solution of the 4 th order step response is shown in Figure 9a.

The poles (denominator roots) of the Butterworth filter are uniformally spaced along a semicircle in the p plane as shown in the Appendix. The numerator of the transfer functions is a constant. Because the root locations are known the transfer function may be directly determined in factored form for any order, n. The designer must choose the order n and the cut-off frequency, ω_c, that will satisfy his requirements.

The Chebyshev filter approximates the ideal gain-frequency characteristic with a maximum pass band error less than a specifiable value. See Figure 5c. By allowing this error to be relatively large, (but still less than the 3 db of the Butterworth filter) a stop band characteristic superior to that of the Butterworth can be achieved. Whereas the Butterworth design philosophy weighs heavily the near zero frequency behavior, the Chebyshev philosophy weighs equally the behavior at all frequencies in the pass band, resulting in a matching of the desired gain at non-uniformly spread discrete frequencies.

The Chebyshev transient response tends to be more oscillatory than the Butterworth. The step response also has greater overshoot. Thus, the Chebyshev filter design (conformally mapped) is less likely to be employed for low pass filter applications than for narrow band pass filter applications where transient response is unimportant.

The poles of the Chebyshev filter are directly related to those of the Butterworth through a simple transformation shown in the Appendix. The designer has three parameters to select n, ω_c, and ϵ.

Transient Response Designs

An ideal impulse response is generally considered to be Gaussian in form: i.e. approximating a normal probability density curve. The corresponding step response approximates the normal cumulative probability curve. (See Figure 6a) This response of course has no overshoot. A figure of merit is the ratio of dispersion time (standard deviation), T_s, to delay time (mean), $\frac{1}{\omega_D}$. For the step response this figure of merit, $\omega_D T_s$, is

* Seifert, W. W. and Steeg, C. W. *Control Systems Engineering,* McGraw-Hill, New York, 1960. P. 636-639.

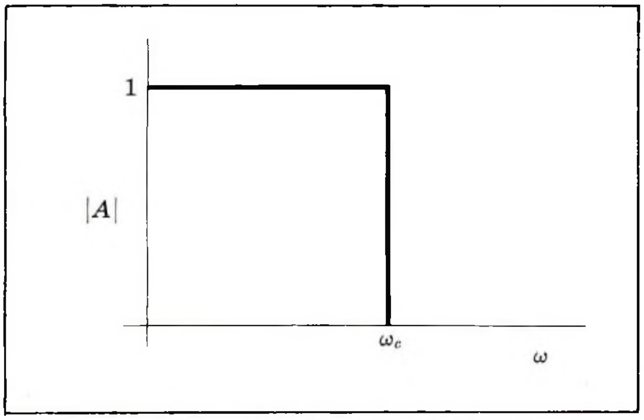


Fig. 5a. Ideal Low Pass Filter

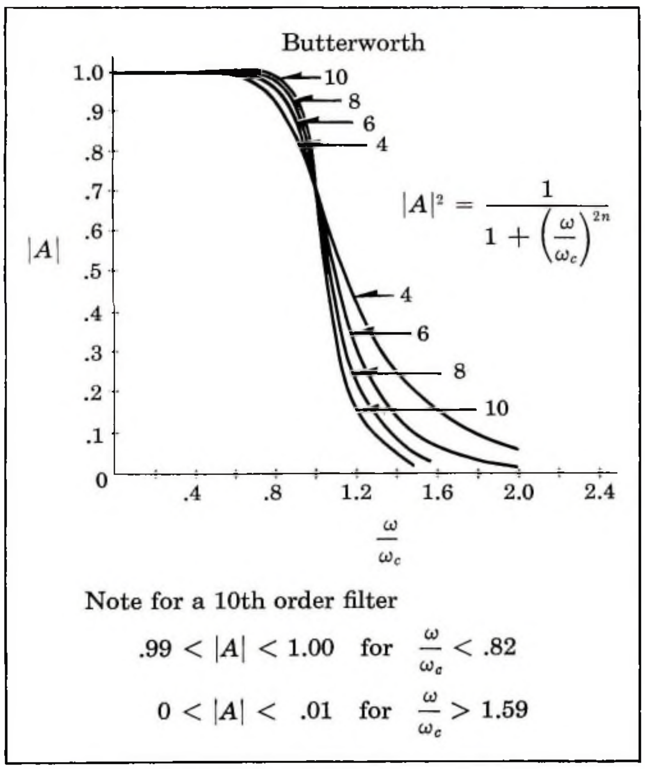


Fig. 5b. Butterworth Low Pass Filter Gain vs. Frequency Characteristic

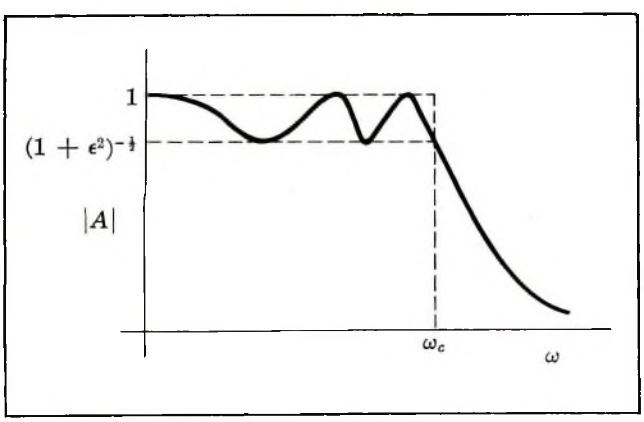


Fig. 5c. Chebyshev Approximation to Low Pass Filter

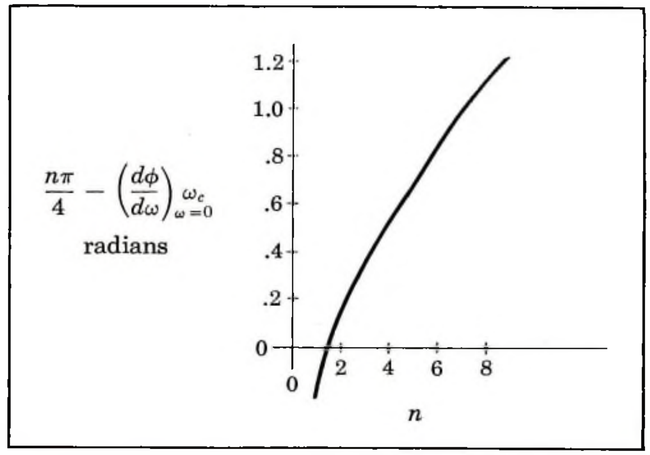


Fig. 5d. Phase Departure from Linear Phase at ω_c vs. n for the Butterworth Filter

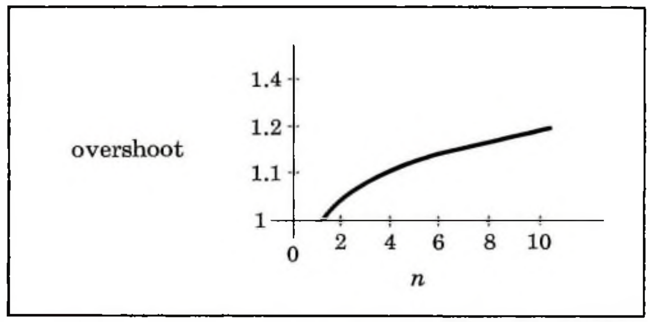


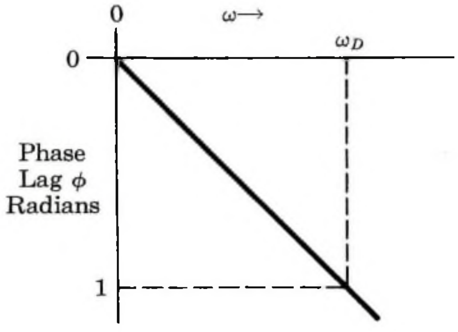
Fig. 5e. Maximum Overshoot, Step Response vs. n for the Butterworth Filter

the ratio of the difference to the sum of the 84 per cent and 16 per cent response times. Thus a Gaussian filter with a small ratio will have a step response with a small rise time compared with delay time. Also the impulse response will be more sharply peaked. The Gaussian characteristics are shown in Figure 6a. Note that the phase vs. frequency characteristic is linear. $\left(\phi = \frac{\omega}{\omega_D}\right)$ Also the dispersion to delay time ratio is a measure of the attenuation in the "pass-band".

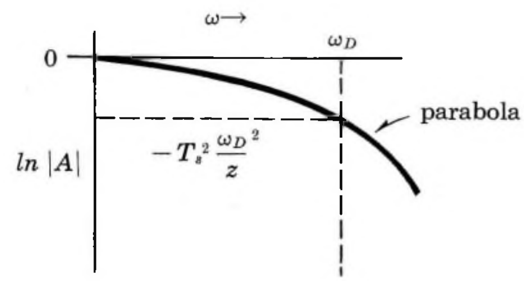
$$\ln |A| = -\tfrac{1}{2} T_s^2 \omega^2$$

The Bessel (or Thomson) filter approximates the ideal phase vs. frequency characteristic in a manner analogous to the Butterworth approximation of the ideal gain vs. frequency characteristic. The nth order Bessel filter matches the ideal phase (zero) and its first $2n$—1 derivatives (all zero except first) with respect to frequency at zero frequency. The third and higher order filters have a nearly Gaussian step response with dispersion to delay time ratio of $\frac{1}{\sqrt{2n-1}}$ See Figure 6b.

The gain vs. frequency characteristic of the Bessel filter is a relatively poor approximation of the ideal rectangular characteristic. However, it is far superior to a cascade of n equal lags which also is approximately Gaussian when n is large. The dispersion to delay time ratio for n equal lags is $\frac{1}{n}$, which is always greater than that of the Bessel. Hence, the Bessel has

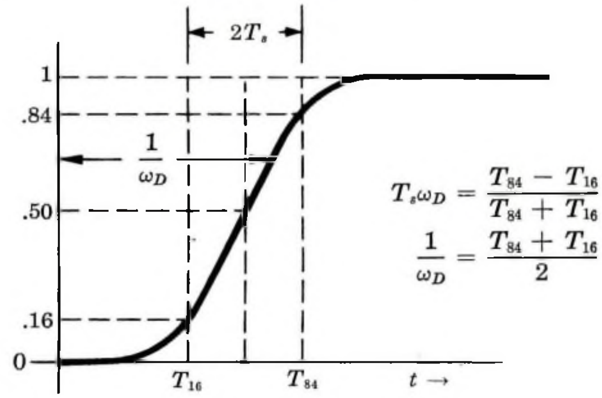


Phase vs. Frequency



Log Gain vs. Frequency

transient response	probability distribution
dispersion time T_s ↔	standard deviation, σ
mean delay time $\frac{1}{\omega_D}$ ↔	mean, m.



Step Response

$$A(s) = e^{-(s/\omega_D) + (T_s^2)(s^2/2)}$$

impulse response

$$h(t) \approx \frac{1}{T_s\sqrt{2\pi}}\, exp\left\{-\tfrac{1}{2}\left[\frac{t}{T_s} - \frac{1}{\omega_D T_s}\right]^2\right\} \text{ for } t > 0$$

$$h(t)_{\max} \approx \frac{.40}{T_s} \quad @\ t = \frac{1}{\omega_D}$$

$$h(t) \approx \frac{.24}{T_s} \quad @\ t = \frac{1}{\omega_D} \pm T_s$$

Fig. 6a. Ideal Low Pass Filter Characteristic (Gaussian)

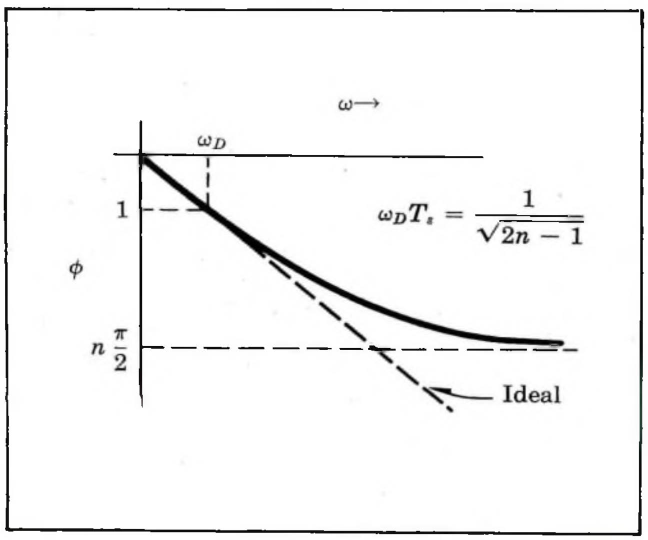


Fig. 6b. Bessel Filter Phase vs. Frequency

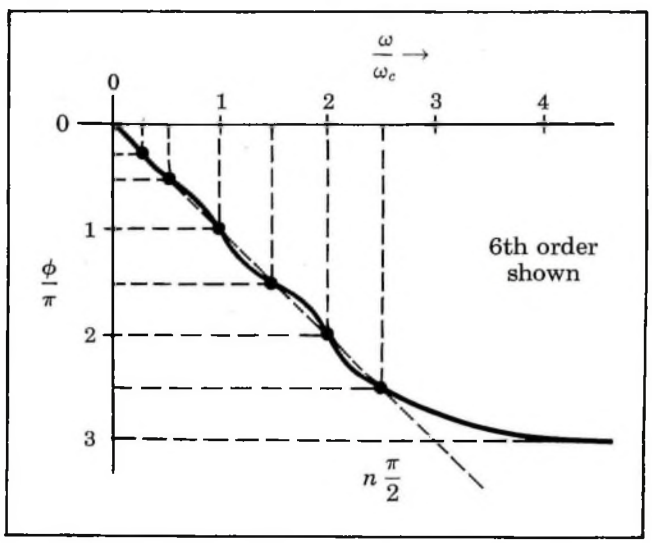


Fig. 6c. Paynter Filter Phase vs. Frequency

less attenuation at low frequencies. Furthermore, at high frequencies the attenuation of the Bessel filter is significantly greater than that of the same order lag filter. Thus, the Bessel more closely approximates the ideal rectangular gain vs. frequency characteristic.

The Bessel transfer function of any order, though difficult to factor, is relatively easy to determine using an iterative procedure, as outlined in the appendix, where the third, fourth, and sixth-order poles are tabulated.

The Paynter filter* employs a philosophy related to that of the Chebyshev filter in approximating the ideal linear phase vs. frequency characteristic. The ideal phase angle $\left(\phi = \pi \frac{\omega}{\omega_c}\right)$ is matched exactly at specific frequencies (where $\phi = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \cdots, (n-1)\frac{\pi}{2}$ radians) The third and higher order filters have a nearly Gaussian transient response with a dispersion time of $T_s = \frac{1}{\sqrt{n-1}} \frac{2}{\omega_c}$ and delay time of $\frac{\pi}{\omega_c}$. The dispersion to delay time ratio, $\frac{2}{\pi} \frac{1}{\sqrt{n-1}}$, is less than that of the Bessel filter for $n \geqslant 4$ thus leading to somewhat less attenuation in the pass-band. Also the attenuation at high frequencies is greater than that of the Bessel filter, leading to an improved stop-band characteristic. Nevertheless, the gain vs. frequency characteristic is quite inferior to that of the same order Butterworth, particularly for large n.

Comparison of Filter Performance

In Figure 8, the 4th-order Butterworth, Bessel, Paynter, and Equal Lag filters are compared with respect to their gain and phase vs. frequency characteristics. The transfer functions are normalized so that unity appears as the coefficient of both the first and last denominator terms. The very high frequency attenuations

$$\lim_{\omega \to \infty} |A| = \frac{1}{\omega_n}$$

as well as the zero frequency gains (unit) are identical. A figure of merit particularly significant when the filter is part of a closed (feedback) loop is the coefficient of the first order denominator term, (a_i), of the normalized transfer function since this defines the low frequency phase lag vs. frequency characteristics $\phi = a_1\omega$ corresponding to a specific high frequency gain characteristic. The Butterworth and Paynter filters are virtually identical in this characteristic and superior to the Bessel and Equal Lag filters for all n.

In Figure 9, the 4th-order step responses are compared. The same basis for normalization was employed.

Averaging Filter

Another type of filter of practical significance results when imaginary zeros are combined with the Paynter filter denominator. The result is a combined low pass and notch filter which approximates the finite time averaging process:

$$y(t) = \frac{1}{T}\int_{t-T}^{t} \times (t)\, dt$$

This filter may be particularly useful in averaging non-stationary random signals and in smoothing commutated signals before sampling. The frequency characteristics are shown in Figure 10 and transient responses in Figure 9 for the third order filter. Design data is included in the appendix. Still another type of filter, the all pass filter, can be developed by employing complex zeros with negative real parts together with the basic Paynter denominator. The numerator terms are made identical in magnitude to the corresponding denominator.

Conclusion

The filter having the most desirable amplitude-frequency response is the Butterworth. The filter having the most desirable transient response is the Paynter. Any even-order implementation of these filters may be economically accomplished by cascading the required number of quadratic, single-amplifier stages.

The Lightning Empiricist, Vol. 11, No. 3, July 1, 1963.
Transfer functions and design data for the Paynter filter are included in the appendix.

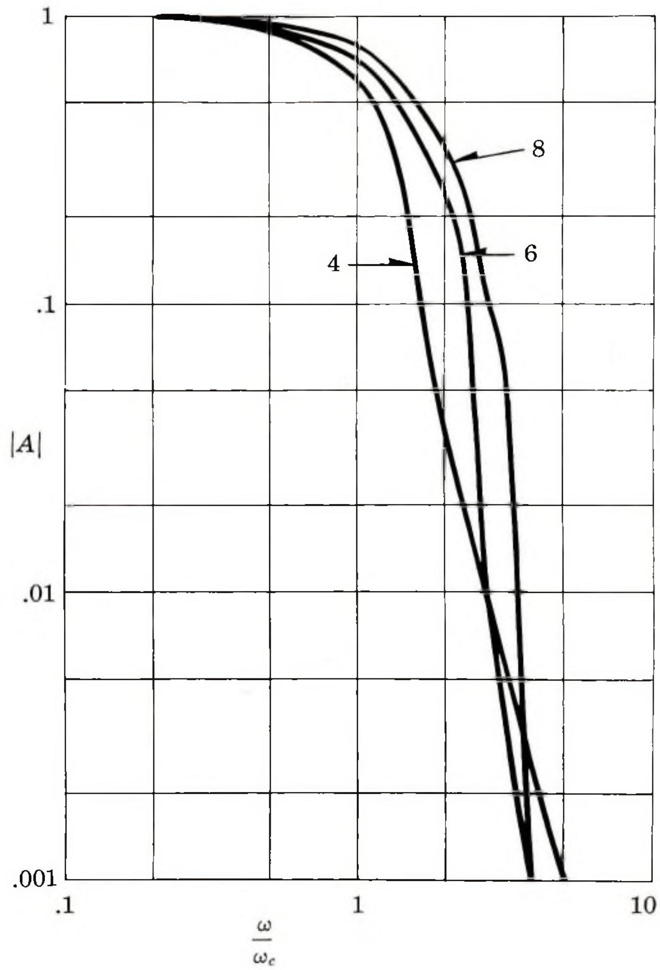


Fig. 7 Amplitude vs. Frequency for 4th, 6th, and 8th Order Paynter Filters

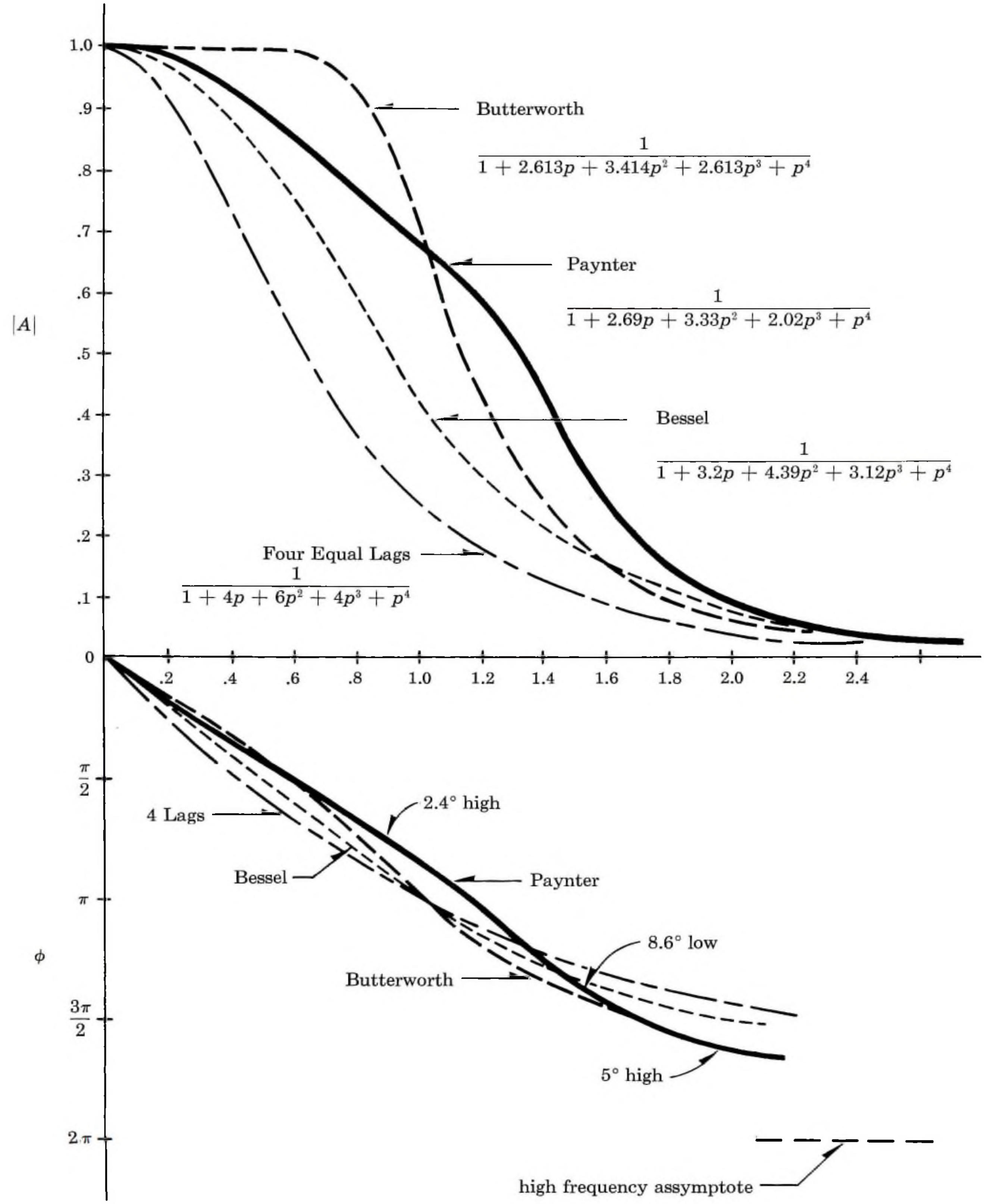


Fig. 8. Comparison of Fourth Order Filters.

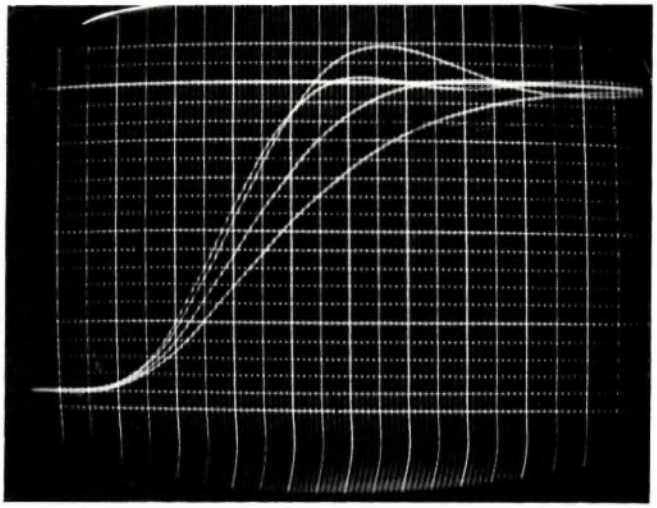


Fig 9a Response of the 4th Order Butterworth, Bessel, Paynter, and Lag Filters to a Step

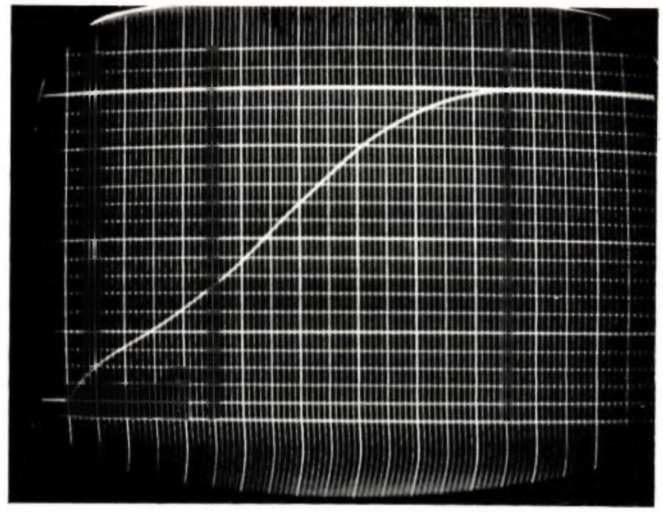


Fig. 9b Response of a 3rd Order "Time Average" Filter to a Step

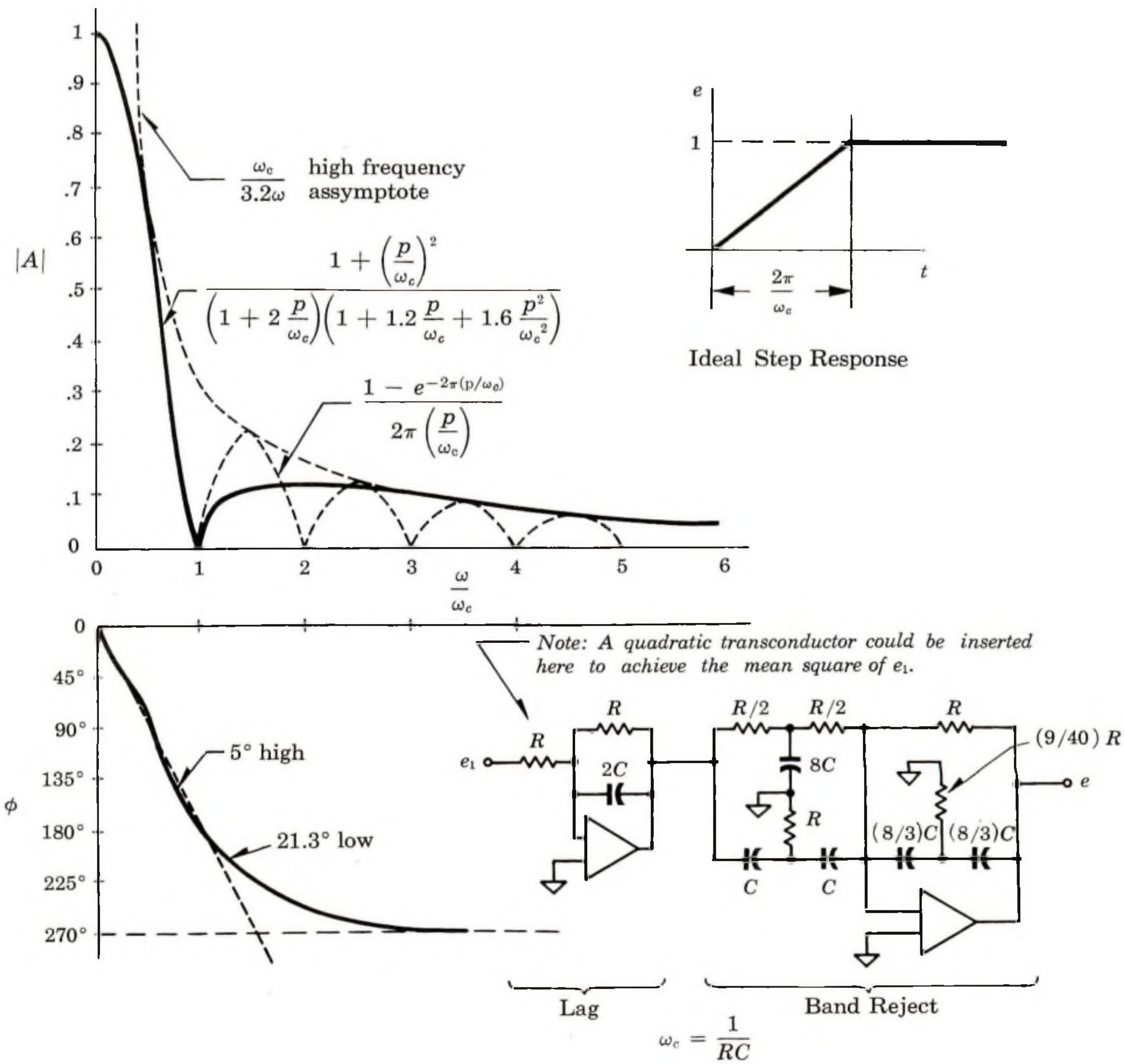
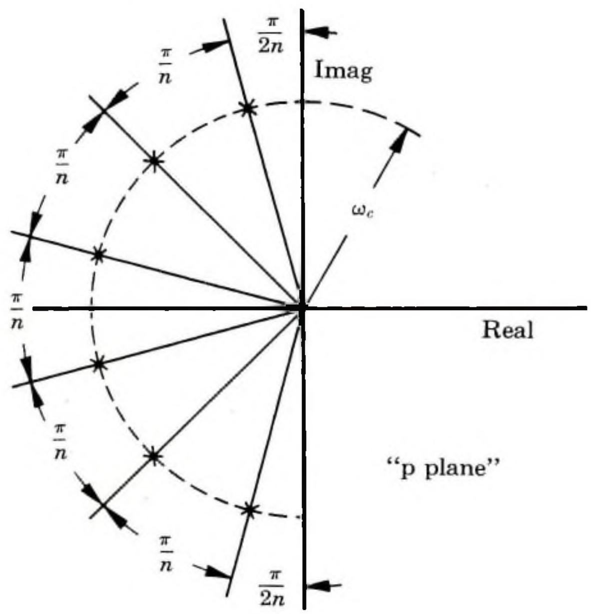


Fig. 10. Gain and Phase vs. Frequency 3rd-Order Finite Time Averaging Filter.

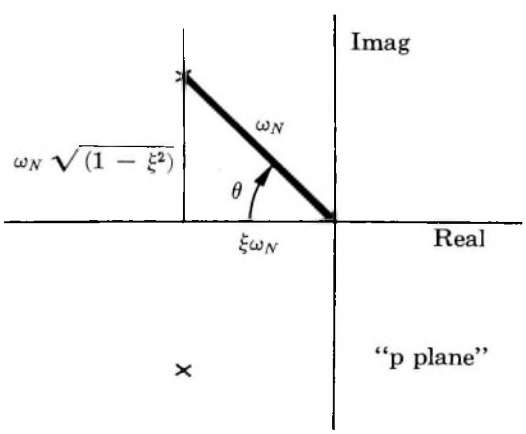
Design Data for Theoretical Filter Types

A. Butterworth Filter

1. Root Location



2. Quadratic Factors from Roots



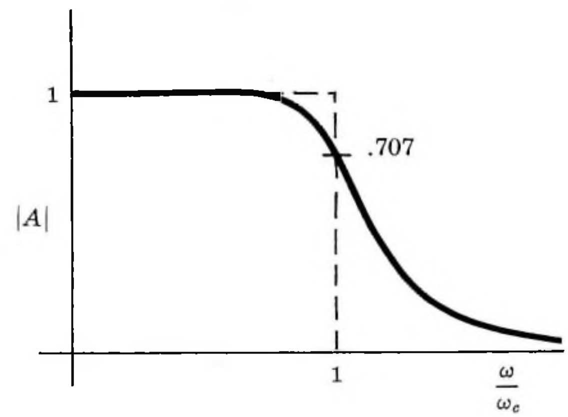
$$\xi = \cos\theta, \quad Q = \frac{1}{2\xi}$$

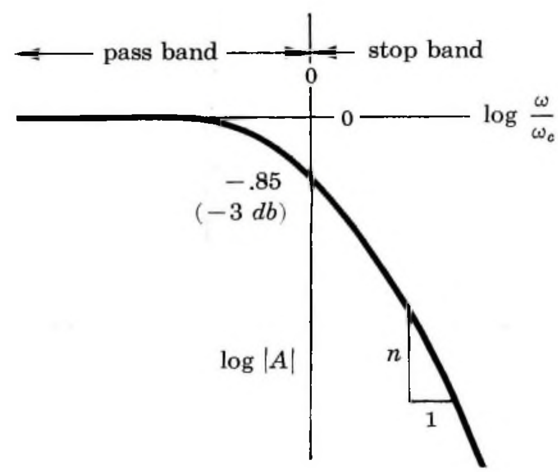
$$A\{p\} = \frac{1}{1 + 2\xi\left(\frac{p}{\omega_N}\right) + \left(\frac{p}{\omega_N}\right)^2}$$

3. Stop Band Characteristic

$$|A| \approx \left(\frac{\omega_c}{\omega}\right)^n \quad \text{for} \quad \frac{\omega}{\omega_c} \gg 1$$

$$\left.\begin{aligned} |A| &= .707 \\ \phi &= n\frac{\pi}{4} \end{aligned}\right\} \quad \text{for} \quad \frac{\omega}{\omega_c} = 1$$





4. Phase for $\frac{\omega}{\omega_c} \ll 1$;

$$\phi = \frac{\omega}{\omega_c} 2 \sum_{i=1}^{n/2} \xi_i = \frac{\omega}{\omega_c} \sum_{i=1}^{n} \cos\theta_i$$

$$\phi \approx 2.613\left(\frac{\omega}{\omega_c}\right) \quad n = 4$$

$$\phi \approx 3.864\left(\frac{\omega}{\omega_c}\right) \quad n = 6$$

$$\phi \approx 5.126\left(\frac{\omega}{\omega_c}\right) \quad n = 8$$

B. Chebyshev Filter

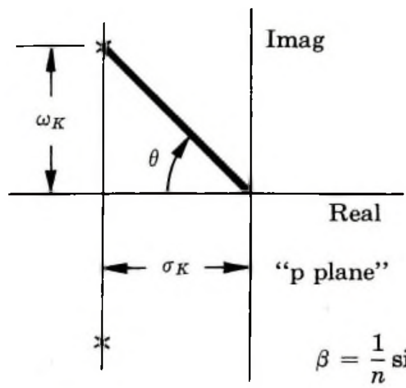
1. Root Location — Related to Butterworth Roots

$(\sigma_K)_{\text{Chebyshev}} = \sinh \beta (\sigma_K)_{\text{Butterworth}}$

$(\omega_K)_{\text{Chebyshev}} = \cosh \beta (\omega_K)_{\text{Butterworth}}$

$(\sigma_K)_{\text{Butterworth}} = \omega_c \cos \theta_K$

$(\omega_K)_{\text{Butterworth}} = \omega_c \sin \theta_K$



$$\beta = \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon}$$

2. Quadratic Factors

$$A(p) = \frac{\sigma_K^2 + \omega_K^2}{p^2 + 2\sigma_K p + \sigma_K^2 + \omega_K^2}$$

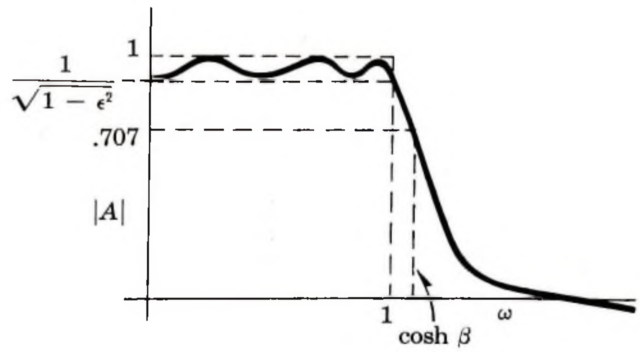
3. Minimum amplitude in pass-band

$$\frac{1}{\sqrt{1+\epsilon^2}}$$

4. Stop-band Characteristic

$$|A| \approx \frac{1}{\epsilon 2^{n-1}} \left(\frac{\omega_c}{\omega}\right)^n \quad \text{for} \quad \frac{\omega}{\omega_c} \gg 1$$

$$|A| = .707 \quad \text{when} \quad \frac{\omega}{\omega_c} = \cosh \beta$$



C. Even Order Paynter Filter

1. Transfer function n even

$$A(p) = \frac{1}{\left\{\left[1+4\left(\frac{p}{\omega_c}\right)^2\right]\left[1+\frac{4}{9}\left(\frac{p}{\omega_c}\right)^2\right]\cdots\left[1+\left(\frac{2}{n-1}\right)^2\left(\frac{p}{\omega_c}\right)^2\right]\right\} + \left\{a_1\left(\frac{p}{\omega_c}\right)\left[1+\left(\frac{p}{\omega_c}\right)^2\right]\cdots\left[1+\left(\frac{2}{n-2}\right)^2\left(\frac{p}{\omega_c}\right)^2\right]\right\}}$$

$$(a_1)_n = (a_1)_{n-2} \frac{1 - \left[\frac{1}{2(n-1)}\right]^2}{1 - \left[\frac{1}{2(n-2)}\right]^2}, \quad (a_1)_2 = 3 \qquad \lim_{n\to\infty} (a_1)_n = \pi$$

2. Phase Lag, ϕ Gain, $|A|$

$$\phi = \pi \frac{\omega}{\omega_c} \quad \text{when} \quad \frac{\omega}{\omega_c} = 0, \tfrac{1}{4}, \tfrac{1}{2}, 1, \cdots \frac{n-1}{2}$$

$$\text{Stop Band} \quad |A| = \frac{1\cdot3\cdot5\ldots(n-1)}{2^{n/2}} \left(\frac{\omega_c}{\omega}\right)^n \quad \text{for} \quad \frac{\omega}{\omega_c} \gg 1$$

3. Fourth and Sixth Order Filters

$$A(p) = \frac{1}{1 + \frac{28}{9}\left(\frac{p}{\omega_c}\right) + \frac{40}{9}\left(\frac{p}{\omega_c}\right)^2 + \frac{28}{9}\left(\frac{p}{\omega_c}\right)^3 + \frac{16}{9}\left(\frac{p}{\omega_c}\right)^4}$$

$$A(p) = \frac{1}{1 + \frac{704}{225}\left(\frac{p}{\omega_c}\right) + \frac{1036}{225}\left(\frac{p}{\omega_c}\right)^2 + \frac{176}{45}\left(\frac{p}{\omega_c}\right)^3 + \frac{112}{45}\left(\frac{p}{\omega_c}\right)^4 + \frac{176}{225}\left(\frac{p}{\omega_c}\right)^5 + \frac{64}{225}\left(\frac{p}{\omega_c}\right)^6}$$

4. Factored Forms:

2nd Order

$$A(p) = \frac{1}{1 + 3\left(\frac{p}{\omega_c}\right) + 4\left(\frac{p}{\omega_c}\right)^2}$$

4th Order

$$A(p) = \frac{1}{\left[1 + 2.62\left(\frac{p}{\omega_c}\right) + 2.42\left(\frac{p}{\omega_c}\right)^2\right]\left[1 + 0.49\left(\frac{p}{\omega_c}\right) + .734\left(\frac{p}{\omega_c}\right)^2\right]}$$

6th Order

$$A(p) = \frac{1}{\left[1 + 2.386\left(\frac{p}{\omega_c}\right) + 1.866\left(\frac{p}{\omega_c}\right)^2\right]\left[1 + .6204\left(\frac{p}{\omega_c}\right) + .6579\left(\frac{p}{\omega_c}\right)^2\right]\left[1 + .1224\left(\frac{p}{\omega_c}\right) + .23175\left(\frac{p}{\omega_c}\right)^2\right]}$$

D. Bessel Filter

1. Denominator Polynomial Function of s, $F_n(p)$

   Normalized for $\omega_D = 1$.

$$F_0(p) = 1$$
$$F_1(p) = p + 1$$
$$F_2(p) = p^2 + 3p + 3$$
$$F_3(p) = p^3 + 6p^2 + 15p + 15$$
$$F_4(p) = p^4 + 10p^3 + 45p^2 + 105p + 105$$
$$F_5(p) = p^5 + 15p^4 + 105p^3 + 420p^2 + 945p + 945$$
$$F_6(p) = p^6 + 21p^5 + 210p^4 + 1260p^3 + 4725p^2 + 10395p + 10395$$
$$\vdots$$
$$F_n(p) = p^2F_{n-2} + (2n-1)F_{n-1}$$

2. Denominator Factors ($\omega_D = 1$)

$$F_3(p) = (p + 2.32219)(p + 1.83891 \pm j\,1.75438)$$
$$F_4(p) = (p + 2.89621 \pm j\,0.867234)(p + 2.10379 \pm j\,2.65742)$$
$$F_6(p) = (p + 4.24836 \pm j\,0.86751)(p + 3.73571 \pm j\,2.62627)(p + 2.5159 \pm j\,4.49267)$$

3. Pass Band Characteristic $\frac{\omega}{\omega_D} < 1$

$$\phi \approx \frac{\omega}{\omega_D} \qquad |A| \approx 1 - \frac{1}{2(2n-1)}\left(\frac{\omega}{\omega_D}\right)^2 + \cdots$$

4. Stop Band Characteristic $\frac{\omega}{\omega_D} \gg 1$

$$|A| \approx [1\cdot3\cdot5\cdot7\cdot\cdots\cdot(2n-1)]\left(\frac{\omega_D}{\omega}\right)^n$$

E. Time Averaging Filter

1. Ideal Transfer Function

$$A(p) = \frac{1 - e^{-Tp}}{Tp}$$

2. Approximate Transfer Function, $T = \frac{2\pi}{\omega_c}$, n even

$$A(p) = \frac{\left[1 + \left(\frac{p}{\omega_c}\right)^2\right]\left[1 + \frac{1}{4}\left(\frac{p}{\omega_c}\right)^2\right]\cdots\left[1 + \left(\frac{2}{n-2}\right)^2\left(\frac{p}{\omega_c}\right)^2\right]}{\text{Paynter Filter Denominator (Order two higher than numerator)}}$$

3. Phase Lag, ϕ

$$\phi = \pi\frac{\omega}{\omega_c} \quad \text{when} \quad \frac{\omega}{\omega_c} = 0, \tfrac{1}{4}, \tfrac{1}{2}, \cdots, \frac{n-1}{2}$$

4. Gain

$$|A| = 1 \quad \text{when} \quad \frac{\omega}{\omega_c} = 0 \qquad |A| = 0 \quad \text{when} \quad \frac{\omega}{\omega_c} = 1, 2, \cdots \left(\frac{n}{2} - 1\right)$$

$$|A| = \frac{1\cdot3\cdot5\ldots(n-1)}{1\cdot2\cdot3\ldots\left(\frac{n}{2}-1\right)2^{n/2}}\left(\frac{\omega_c}{\omega}\right)^2 \quad \text{for} \quad \frac{\omega}{\omega_c} \gg \left(\frac{n}{2} - 1\right)$$

$$|A| < \frac{1}{\pi}\left(\frac{\omega_c}{\omega}\right) \quad \text{for all } \omega$$

5. Third Order Transfer Function, Factored Form.

$$A(p) = \frac{1 + \left(\frac{p}{\omega_c}\right)^2}{\left(1 + 2\frac{p}{\omega_c}\right)\left[1 + 1.2\left(\frac{p}{\omega_c}\right) + 1.6\left(\frac{p}{\omega_c}\right)^2\right]}$$

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